Izvestiya: *Mathematics* **70**:6 1225–1232

Izvestiya RAN: *Ser. Mat.* **70**:6 153–160 DOI 10.1070/IM2006v070n06ABEH002344

On the mean value of the ladder epoch for random walks with a small drift

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Abstract. In this paper, we prove theorems on the asymptotic behaviour of the mean value of the time at which the non-negative half-axis is first attained for a random walk whose drift tends to zero. It is assumed that the distribution of jumps of the random walk belongs to the domain of attraction of a stable law with exponent $\alpha \in (1,2)$.

Let $\{\xi_n^0\}$, $n \geqslant 1$, be a sequence of independent identically distributed random variables, let $\mathbf{E}\,\xi_1^0 = 0$, let $F(x) = \mathbf{P}(\xi_1^0 < x)$ and let

$$F(-x) = \frac{a + o(1)}{x^\alpha}\, L(x), \qquad 1 - F(x) = \frac{b + o(1)}{x^\alpha}\, L(x) \tag{1}$$

as $x \to \infty$, where $L(x)$ is a slowly varying function, $1 < \alpha < 2$, $a \geqslant 0$, $b \geqslant 0$ and $a + b > 0$.

It is known (for example, see [1]) that the above condition is equivalent to the requirement that the distribution F should belong to the domain of attraction of a stable distribution F_α, that is, the relation

$$\mathbf{P}\biggl(\frac{S_n^0}{B_n} - A_n < x\biggr) \to F_\alpha(x) \tag{2}$$

should hold for the sequence $S_n^0 = \xi_1^0 + \dots + \xi_n^0$, $n \geqslant 1$, where A_n and B_n are suitable number sequences. It can easily be seen that one can set $A_n = 0$ for $\mathbf{E}\,\xi_1^0 = 0$, and then the stable limit law will also have zero mean.

Generally speaking, stable distributions form a four-parameter family. Let $\psi(t)$ be the logarithm of the characteristic function of a stable distribution. Various representations are known for $\psi(t)$; in what follows, it will be convenient to use the following one (see [1]):

$$\psi(t) = it\gamma - \lambda |t|^\alpha \exp\biggl\{-i \frac{\pi t}{2|t|}\, \beta(2-\alpha)\biggr\} \tag{3}$$

for $\alpha \in (1,2)$. Here $-1 \leqslant \beta \leqslant 1$, $\lambda > 0$ and γ is an arbitrary real number. In the case under consideration we have $\gamma = 0$ and, without loss of generality, λ can

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This work was supported by the Russian Foundation for Basic Research (project no. 05-01-00810) and the Programme for the support of Leading Scientific Schools of the Russian Federation (grant no. NSh-8980.2006.1).

AMS 2000 *Mathematics Subject Classification.* 11K05, 11N60, 40E05, 26A12, 60E07, 60E10, 60F05, 60G50.

be set equal to unity. The number β is found from (1) in the well-known way. In what follows, we shall use the notation $F_\alpha = F_{\alpha,\beta}$.

Let $\xi_n = \xi_n^0 + \varepsilon$ and $S_n = S_n^0 + n\varepsilon$, where $\varepsilon > 0$ is a small number. We introduce the ladder epochs

$$\eta_- = \inf\{n \geqslant 1 \colon S_n < 0\}, \qquad \eta_+ = \inf\{n \geqslant 1 \colon S_n \geqslant 0\}$$

(here we assume that $\eta_+ = \infty$ if $S_n < 0$ for all n and $\eta_- = \infty$ if $S_n \geqslant 0$ for all n) and the ladder heights $\chi_\pm = S_{\eta_\pm}$. The quantity χ_- will be assumed indeterminate for the event $\{\eta_- = \infty\}$. We write $q = \mathbf{P}(\eta_- < \infty)$. It is known that $\mathbf{E}\,\eta_+ = (1-q)^{-1}$ (the proof of Theorem 1 below contains a justification of this relation). The expression $\mathbf{E}\,\eta_+$ may increase indefinitely as $\varepsilon \to 0$, namely, $\mathbf{E}\,\eta_+ = \infty$ if $\mathbf{E}\,\xi_1 = 0$. (For a discussion of this situation see [2], where an upper bound for $\mathbf{E}\,\eta_+$ is established under more general conditions than those in the present paper.) For the scheme under consideration, the result in [2] reads as follows: for an arbitrarily small $\delta > 0$, there is a constant $C = C(\delta)$ such that $\mathbf{E}\,\eta_+ \leqslant C/\varepsilon^{\theta-1+\delta}$, where we write $\theta = \alpha/(\alpha-1)$ for brevity. [2] also contains the easily derived lower bound

$$\mathbf{E}\,\eta_+ \geqslant \frac{\mathbf{E}\,\xi_1^+}{\varepsilon}\,,$$

where $\xi_1^+ = \max(0, \xi_1)$.

The object of this paper is to investigate the asymptotic behaviour of $\mathbf{E}\,\eta_+$ as $\varepsilon \to 0$ under the above conditions. We write $p = F_{\alpha,\beta}(0)$.

Theorem 1. *Let condition* (1) *hold. Then*

$$\mathbf{E}\,\eta_+ = \varepsilon^{-\theta p(1+o(1))}, \qquad \varepsilon \to 0.$$

Remark 1. Let $\gamma = 0$ and $\lambda = 1$ in the representation (3). In this case the value of p is known [3]:

$$p = \frac{1}{2}\biggl(1 - \frac{\beta(2-\alpha)}{\alpha}\biggr).$$

It follows at once that $\alpha p \leqslant 1$ since $2(1-\alpha p) = (1+\beta)(2-\alpha) \geqslant 0$. Thus, $\theta p \leqslant \theta - 1$, and this shows that, generally speaking, the upper bound given in [2] overestimates the growth rate.

Proof of Theorem 1. Let

$$R_\pm(z,\mu) = 1 - \mathbf{E}\bigl(z^{\eta_\pm}\exp\{\mu\chi_\pm\};\ \eta_\pm < \infty\bigr)$$

for $|z| \leqslant 1$ and $\operatorname{Re}\mu = 0$. The functions $R_\pm(z,\mu)$ are components of the well-known factorization (for example, see [4])

$$1 - z\varphi(\mu) = R_+(z,\mu)R_-(z,\mu),$$

where $\varphi(\mu) = \mathbf{E}\,e^{\mu\xi_1}$. Setting $z = 1$ in this identity, we arrive at the relation

$$R_-(1,\mu) = \frac{1-\varphi(\mu)}{R_+(1,\mu)}\,.$$

As $\mu \to 0$, we use Wald's identity to obtain the relations

$$1 - q = \frac{\varepsilon}{\mathbf{E}\,\chi_+} = (\mathbf{E}\,\eta_+)^{-1}.$$

Next, for $|z| < 1$ and $\operatorname{Re}\mu = 0$, we have the well-known representation

$$R_-(z, \mu) = \exp\biggl\{-\sum_{n=1}^{\infty} \frac{z^n}{n}\,\mathbf{E}\bigl(\exp\{\mu S_n\};\ S_n < 0\bigr)\biggr\}.$$

Therefore

$$\begin{aligned}\mathbf{E}\,\eta_+ &= (1-q)^{-1} = R_-^{-1}(1,0)\\ &= \lim_{z\to 1} \exp\biggl\{\sum_{n=1}^{\infty} \frac{z^n}{n}\,\mathbf{P}(S_n < 0)\biggr\} = \exp\biggl\{\sum_{n=1}^{\infty} \frac{\mathbf{P}(S_n^0 < -n\varepsilon)}{n}\biggr\}.\end{aligned} \tag{4}$$

Thus, it is necessary to consider the asymptotic series

$$\sum_{n=1}^{\infty} \frac{\mathbf{P}(S_n^0 < -n\varepsilon)}{n}$$

as $\varepsilon \to 0$. It is established in [5] that, under the assumptions of Theorem 1, the asymptotic formula

$$\sum_{n=1}^{\infty} \frac{\mathbf{P}\bigl(|S_n^0| > n\varepsilon\bigr)}{n} = \frac{\alpha}{\alpha - 1}\,|\log\varepsilon|\bigl(1 + o(1)\bigr), \qquad \varepsilon \to 0, \tag{5}$$

holds. Passing to a consideration of one-sided deviations $\mathbf{P}(S_n^0 < -n\varepsilon)$ instead of two-sided ones hardly changes the proof in [5], and therefore we shall not repeat it here, merely noting that the proof uses the representation

$$\sum_{n=1}^{\infty} \frac{1}{n}\,\mathbf{P}(S_n^0 < -n\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon), \tag{6}$$

where

$$\begin{aligned}I_1(\varepsilon) &= \sum_{n=1}^{\infty} \frac{1}{n}\biggl(\mathbf{P}(S_n^0 < -n\varepsilon) - F_{\alpha,\beta}\Bigl(-\frac{n\varepsilon}{B_n}\Bigr)\biggr),\\ I_2(\varepsilon) &= \sum_{n=1}^{\infty} \frac{1}{n}\,F_{\alpha,\beta}\Bigl(-\frac{n\varepsilon}{B_n}\Bigr).\end{aligned}$$

Using a string of estimates, it is established that

$$\begin{gathered}I_1(\varepsilon) = o(|\log\varepsilon|),\\ I_2(\varepsilon) = \frac{\alpha p}{\alpha - 1}\,|\log\varepsilon|\bigl(1 + o(1)\bigr), \qquad \varepsilon \to 0.\end{gathered}$$

The last relation is obtained using the Euler–Maclaurin formula. The same technique is used below in the proof of Theorem 2, which differs from the proof of Theorem 1 in that the estimation of $I_1(\varepsilon)$ is simpler (owing to the additional condition imposed in Theorem 2) and the investigation of the asymptotic expression for $I_2(\varepsilon)$ is more detailed. The theorem is proved.

Theorem 1 describes the asymptotic behaviour of $\log \mathbf{E}\,\eta_+$. In what follows, we shall show that, under certain additional conditions, this result can be strengthened. Let us write

$$\Delta_n = \sup_x \left| \mathbf{P}\left(\frac{S_n^0}{B_n} < x \right) - F_{\alpha,\beta}(x) \right|.$$

To derive the estimate $I_1(\varepsilon) = o(|\log \varepsilon|)$ in the proof of Theorem 1, use was made in [5] of the property that $\Delta_n \to 0$ as $n \to \infty$. To obtain a stronger result on the asymptotic behaviour of $\mathbf{E}\,\eta_+$, it is required to impose a condition on the rate of convergences of Δ_n to zero. There are some papers in which various sufficient conditions are established to ensure that Δ_n decreases at the required rate. For example, in [6], conditions on F are stated in terms of second-order regularity. Under these conditions, Δ_n is a regularly varying function of the form $\Delta_n = L(n)n^\gamma$ with exponent $\gamma \in [-1, 0]$, where $L(n)$ is a slowly varying function.

Theorem 2. *In addition to the hypotheses of Theorem 1, assume that the series*

$$\sum_{n=1}^{\infty} \frac{\Delta_n}{n} < \infty$$

is convergent. Then there is a function $l(x)$ slowly varying at infinity such that

$$\mathbf{E}\,\eta_+ = \varepsilon^{-\theta p} l(\varepsilon^{-1})\bigl(1 + o(1)\bigr)$$

as $\varepsilon \to 0$. If $L(x) \equiv \mathrm{const}$ in formula (1), then we also have $l(x) \equiv \mathrm{const}$.

An exact expression for $l(x)$ is given at the end of the proof.

Proof. By (4), it suffices to show that there is a slowly varying function $l(x)$ such that the relation

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbf{P}(S_n^0 < -n\varepsilon) = -\theta p \log \varepsilon + \log l(\varepsilon^{-1}) + o(1)$$

holds. We shall use the representation (6). As a consequence of the hypotheses of the theorem, the series

$$I_1(\varepsilon) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\mathbf{P}(S_n^0 < -n\varepsilon) - F_{\alpha,\beta}\left(-\frac{n\varepsilon}{B_n} \right) \right)$$

is convergent uniformly with respect to ε, and therefore

$$\begin{aligned} \lim_{\varepsilon \to 0} I_1(\varepsilon) &= \sum_{n=1}^{\infty} \frac{1}{n} \lim_{\varepsilon \to 0} \left(\mathbf{P}(S_n^0 < -n\varepsilon) - F_{\alpha,\beta}\left(-\frac{n\varepsilon}{B_n} \right) \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \bigl(\mathbf{P}(S_n^0 < 0) - F_{\alpha,\beta}(0) \bigr), \end{aligned}$$

that is, $I_1(\varepsilon) = C_1 + o(1)$. $C, C_1, C_2, \dots$ everywhere denote constants, which may be different in different calculations.

To estimate $I_2(\varepsilon)$, we make use of the arguments in [5].

It is known that the normalizing constants in (2) have the form $B_n = cn^{1/\alpha}L_1(n)$ for an appropriate choice of the constant c, and $L_1(x)$, $L_1(1) = 1$, is a slowly varying function. Put $\delta = 1 - 1/\alpha = 1/\theta$. The convergence in (2) is preserved if $L_1(x)$ is replaced by an arbitrary slowly varying function $L_0(x)$ asymptotically equivalent to $L_1(x)$, that is, $L_1(x)/L_0(x) \to 1$ as $x \to \infty$. Therefore $L_1(x)$ can be assumed differentiable (see [8], § 1.3.2), and $y = x^\delta/L_1(x)$ can be regarded as a monotone function (see [8], § 1.5.3).

Let $\Phi(x) = F_{\alpha,\beta}(-x/c)$. An application of the Euler–Maclaurin formula (see [7], § 12.2) results in the expression

$$I_2(\varepsilon) = \int_1^\infty \frac{1}{x}\,\Phi\biggl(\frac{\varepsilon x^\delta}{L_1(x)}\biggr)\,dx + \frac{1}{2}\,\Phi(\varepsilon) - \int_1^\infty P_1(x)\,d\biggl(\frac{1}{x}\,\Phi\biggl(\frac{\varepsilon x^\delta}{L_1(x)}\biggr)\biggr), \tag{7}$$

where $P_1(x) = [x] - x + 1/2$.

We shall consider the asymptotic behaviour of each of the three terms on the right-hand side of (7). For the first, we make the change of variable $y = x^\delta/L_1(x)$. Then we obtain $x = g(y) = y^{1/\delta}L_2(y)$ (see [8], § 1.5.7), where $L_2(y)$, $L_2(1) = L_1(1) = 1$, is a slowly varying function. Thus, we have

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,\Phi\biggl(\frac{\varepsilon x^\delta}{L_1(x)}\biggr)\,dx &= \int_1^\infty \frac{g'(y)}{g(y)}\,\Phi(\varepsilon y)\,dy = \int_1^\infty \Phi(\varepsilon y)\,d\log g(y)\\ &= \int_\varepsilon^\infty \Phi(t)\,d\log g\biggl(\frac{t}{\varepsilon}\biggr) = -\int_\varepsilon^\infty \Phi'(t)\log g\biggl(\frac{t}{\varepsilon}\biggr)\,dt\\ &= -\frac{1}{\delta}\int_\varepsilon^\infty \Phi'(t)\log t\,dt + \frac{\log\varepsilon}{\delta}\int_\varepsilon^\infty \Phi'(t)\,dt - \int_\varepsilon^\infty \Phi'(t)\log L_2\biggl(\frac{t}{\varepsilon}\biggr)\,dt. \end{aligned}\tag{8}$$

It can easily be seen that

$$\int_\varepsilon^\infty \Phi'(t)\log t\,dt = C_2 + O(\varepsilon), \qquad \int_\varepsilon^\infty \Phi'(t)\,dt = -p + O(\varepsilon)$$

as $\varepsilon \to 0$.

Let us consider the third term on the right-hand side of (8). If $L(x) \equiv \text{const}$, then $L_1(x) \equiv 1$ and $L_2(x) \equiv 1$, and therefore the integral in question is zero. In the general case, we have

$$\begin{aligned}\int_\varepsilon^\infty \Phi'(t)\log L_2\biggl(\frac{t}{\varepsilon}\biggr)\,dt &= \int_\varepsilon^\infty \biggl(\log L_2\biggl(\frac{t}{\varepsilon}\biggr) - \log L_2\biggl(\frac{1}{\varepsilon}\biggr)\biggr)\Phi'(t)dt\\ &\quad + \log L_2\biggl(\frac{1}{\varepsilon}\biggr)\int_\varepsilon^\infty \Phi'(t)\,dt = \int_\varepsilon^\infty \Phi'(t)\log\frac{L_2(t/\varepsilon)}{L_2(1/\varepsilon)}\,dt\\ &\quad + \log L_2\biggl(\frac{1}{\varepsilon}\biggr)\bigl(-p + O(\varepsilon)\bigr). \end{aligned}\tag{9}$$

To estimate the integral on the right-hand side of (9), we use Potter's theorem (see [8], § 1.5.4), according to which there is a constant $C_3 > 1$ such that

$$\frac{L_2(x)}{L_2(y)} \leqslant C_3 \max\left(\frac{x}{y}, \frac{y}{x}\right)$$

for all $x > 0$, $y > 0$. Therefore

$$\left| \int_\varepsilon^\infty \Phi'(t) \log \frac{L_2(t/\varepsilon)}{L_2(1/\varepsilon)}\, dt \right| \leqslant \log C_3 + \int_{0+}^1 \bigl|\Phi'(t)\bigr| \log(t^{-1})\, dt + \int_1^\infty |\Phi'(t)| \log t\, dt < \infty,$$

that is, by the majorized convergence theorem, we have

$$\int_\varepsilon^\infty \Phi'(t) \log \frac{L_2(t/\varepsilon)}{L_2(1/\varepsilon)}\, dt = \int_0^\infty \Phi'(t)\, I_{[\varepsilon,\infty)}(t) \log \frac{L_2(t/\varepsilon)}{L_2(1/\varepsilon)}\, dt \to 0$$

as $\varepsilon \to 0$.

It has thus been established that the relation

$$\int_\varepsilon^\infty \Phi'(t) \log L_2\left(\frac{t}{\varepsilon}\right) dt = \log l_1\left(\frac{1}{\varepsilon}\right) + o(1)$$

holds for the slowly varying function $l_1(x) = L_2^p(x)$.

For the second term on the right-hand side of (7), we clearly have $\Phi(\varepsilon) = p + O(\varepsilon)$. Finally, let us consider the third term on the right-hand side of (7),

$$\int_1^\infty P_1(x)\, d\left(\frac{1}{x}\, \Phi\left(\frac{\varepsilon x^\delta}{L_1(x)}\right)\right) = -\int_1^\infty P_1(x)\, \frac{1}{x^2}\, \Phi\left(\frac{\varepsilon x^\delta}{L_1(x)}\right) dx + \int_1^\infty P_1(x)\, \frac{1}{x}\, d\Phi\left(\frac{\varepsilon x^\delta}{L_1(x)}\right). \tag{10}$$

Since the function $\Phi\bigl(\varepsilon x^\delta / L_1(x)\bigr)$ is bounded uniformly with respect to ε and the integral

$$\int_1^\infty P_1(x) \frac{1}{x^2}\, dx$$

is convergent, the first integral on the right-hand side of (10) is equal to $C_4 + O(\varepsilon)$.

Let us consider the second integral on the right-hand side of (10). After the change of variable $y = x^\delta / L_1(x)$ used above, we obtain

$$\int_1^\infty P_1(x) \frac{1}{x}\, d\Phi\left(\frac{\varepsilon x^\delta}{L_1(x)}\right) = \int_1^\infty P_1\bigl(g(y)\bigr) \frac{1}{g(y)}\, d\Phi(\varepsilon y) = \varepsilon \int_1^\infty P_1\bigl(g(y)\bigr) \frac{1}{g(y)}\, \Phi'(\varepsilon y)\, dy = O(\varepsilon)$$

since $P_1\bigl(g(y)\bigr)\Phi'(\varepsilon y)$ is bounded uniformly with respect to ε, $g(y) = y^{1/\delta} L_2(y)$ and $\delta = 1 - 1/\alpha < 1$.

Hence, as $\varepsilon \to 0$ we have

$$I_1(\varepsilon) + I_2(\varepsilon) = -\theta p \log \varepsilon + \log l\left(\frac{1}{\varepsilon}\right) + o(1),$$

where

$$l(x) = CL_2^p(x), \qquad C = \exp\left\{C_1 - \theta C_2 + \frac{p}{2} + C_4\right\},$$

$$C_1 = \sum_{n=1}^{\infty} \frac{1}{n}\bigl(\mathbf{P}(S_n^0 < 0) - F_{\alpha,\beta}(0)\bigr), \qquad C_2 = \int_0^{\infty} \Phi'(t) \log t \, dt,$$

$$C_4 = p \int_1^{\infty} P_1(t) \frac{1}{t^2} \, dt.$$

After some simple calculations, we obtain

$$C_4 = p\left(C_0 - \frac{1}{2}\right),$$

where $C_0 = 0.5772\ldots$ is Euler's constant, that is, we finally have

$$C = \exp\{C_1 - \theta C_2 + pC_0\}.$$

Recall that the slowly varying function L_2 is found from the relation $x = y^{1/\delta} L_2(y)$, where $y = x^\delta / L_1(x)$. The function L_1 is involved in the definition of the normalizing constants in (2), namely, $B_n = cn^{1/\alpha} L_1(n)$, $L_1(1) = 1$. Here $L_1(x)$ can be assumed differentiable and $y = x^\delta / L_1(x)$ can be regarded as monotone. The theorem is proved.

It is clear that the use of Wald's identity $\mathbf{E}\,\chi_+ = \varepsilon\,\mathbf{E}\,\eta_+$ also enables one to determine an asymptotic expression for $\mathbf{E}\,\chi_+$, and asymptotic estimates can also be found for $\mathbf{E}\,\chi_+^s$. This follows from the inequality

$$\mathbf{E}\,\chi_+^s \leqslant C(\mathbf{E}\,\eta_+)^{s-\alpha_1+1}$$

established in [2] (and used there), which is valid under the conditions of the present paper for any α_1 such that $1 < \alpha_1 < \alpha$ and $\alpha_1 - 1 < s \leqslant \alpha_1$.

The author is grateful to A. A. Borovkov, I. S. Borisov and S. V. Nagaev for valuable discussions.

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Received 21/JAN/05
Translated by V. M. VOLOSOV