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FACTORIZATION IDENTITIES FOR THE SOJOURN TIME OF A RANDOM WALK IN A STRIP

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Abstract: We find factorization representations for the moment generating function of the joint distribution of the sojourn time of a random walk in a strip and half-line in finitely many steps and of the location at the last time moment.

Keywords: random walk, sojourn time in a strip, factorization identities

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent identically distributed random variables, $S_n = X_1 + \dots + X_n$, $n \geq 1$. Given a Borel set $D \subset R$, we introduce

$$T_n = T_n(D) = \sum_{k=1}^{n} I_{\{S_k \in D\}},$$

where $I_A(\omega) = 1$ if $\omega \in A$, and $I_A(\omega) = 0$ otherwise; T_n is the number of points k, $1 \leq k \leq n$, such that $S_k \in D$.

The problem consists in studying the joint distribution of $\mathbf{P}(T_n = k, S_n \in A)$.

We do not survey the numerous articles that are devoted to the case $D = (0, \infty)$ and the arcsine law connected with this case (for instance, see [1]). Of prime interest are the cases of a finite interval $D = [a, b]$ and sets of the form $(-\infty, b]$ or $[a, \infty)$. Throughout the sequel, we assume that $a \leq 0 \leq b$, although this is not principal in view of the obvious relation $T_n(D) = n - T_n(\overline{D})$. We observe that the limit distribution for $T_n([a, b])$ as $n \to \infty$ if a and b are fixed is well known (see [2, 3]). In [4], for $D = [a, b]$, the asymptotic expansions of the probabilities $\mathbf{P}(T_n = k)$ and $\mathbf{P}(T_n \geq k)$ were obtained as $n \to \infty$ and for various constraints on the growth rate of $k = k(n)$ under the following assumptions:

1) X_n is integer-valued;

2) the Cramér condition holds; i.e., $\mathbf{E}|t|^{X_1} < \infty$ for $1 - \delta < |t| < 1 + \delta$, $\delta > 0$.

The main idea of [4] is as follows: The random variable T_n is representable as

$$T_n = \min\{k \geq 1 : Y_1 + \dots + Y_k > n\} - 1,$$

where $Y_1, Y_1 + Y_2, \dots$ are successive time moments when a walking particle stays in $[a, b]$. The random variables $Y_1, Y_2, \dots$ are not identically distributed; the distribution of the span of the time needed for passage from $i \in [a, b]$ to $j \in [a, b]$ depends on i and j. For this reason, the problem reduces to considering a random walk on a Markov chain with the state set $\mathscr{X} = \{a, a+1, \dots, b\}$. Using this approach requires the knowledge of the distribution of Y_k, which is a rather difficult problem not considered in [4]. The asymptotic expansions in [4] are obtained by the factorization method consisting in several stages. At the first stage, the factorization representations are constructed for the double (or even triple) Laplace–Stieltjes transforms of the sought distributions. For a random walk on a Markov chain with the state set $\mathscr{X}$, the dimension of matrices in factorization equals $b - a + 1$, which greatly complicates the further study of probabilities.

Some alternative factorization matrix representation was found later in [5] for the moment generating function of (T_n, S_n) in the case of a finite interval D. However, it turned out that the process of calculating this representation is rather difficult even for semicontinuous random walks (see [6]).

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Just for this reason, we start with posing the problem of finding a moment generating function of a sought distribution without using matrix representation.

Given variables $|z| < 1$, $|uz| < 1$, and $\operatorname{Re}\lambda = 0$, introduce the functions $\varphi(\lambda) = \mathbf{E}e^{\lambda X_1}$,

$$Q_0(z,u,\lambda) = \sum_{n=1}^{\infty} z^n \sum_{k=0}^{n} u^k \int_D e^{\lambda x}\mathbf{P}(T_n = k, S_n \in dx),$$

$$Q(z,u,\lambda) = \sum_{n=1}^{\infty} z^n \sum_{k=0}^{n} u^k \int_{\overline{D}} e^{\lambda x}\mathbf{P}(T_n = k, S_n \in dx).$$

Theorem 1. *The following identity takes place for an arbitrary Borel set $D \subset R$ and the above-indicated values of u, z, and λ:*

$$uQ(z,u,\lambda)(1 - z\varphi(\lambda)) + (1 + Q_0(z,u,\lambda))(1 - zu\varphi(\lambda)) = 1. \tag{1}$$

Proof. Add one more jump of a random walk to each of the trajectories that terminate at time n and satisfy the property $\{T_n = k\}$. Then, at time $n+1$, the random particle either is outside D and in this case $T_{n+1} = T_n = k$ or $S_{n+1} \in D$ and so $T_{n+1} = k+1$. In terms of the Laplace–Stieltjes transforms, the above step can be written as

$$\varphi(\lambda) \int_{-\infty}^{\infty} e^{\lambda x}\mathbf{P}(T_n = k, S_n \in dx)$$
$$= \int_D e^{\lambda x}\mathbf{P}(T_{n+1} = k+1, S_{n+1} \in dx) + \int_{\overline{D}} e^{\lambda x}\mathbf{P}(T_{n+1} = k, S_{n+1} \in dx).$$

Multiply this equality by $u^{k+1}z^{n+1}$, sum over k from 0 to n, and sum over n from 1 to ∞. Then

$$zu\varphi(\lambda)(Q_0(z,u,\lambda) + Q(z,u,\lambda)) = \sum_{n=1}^{\infty} z^{n+1} \sum_{k=0}^{n} u^{k+1} \int_D e^{\lambda x}\mathbf{P}(T_{n+1} = k+1, S_{n+1} \in dx)$$
$$+ \sum_{n=1}^{\infty} z^{n+1} \sum_{k=0}^{n} u^{k+1} \int_{\overline{D}} e^{\lambda x}\mathbf{P}(T_{n+1} = k, S_{n+1} \in dx) = \sum_{m=2}^{\infty} z^m \sum_{i=1}^{m} u^i \int_D e^{\lambda x}\mathbf{P}(T_m = i, S_m \in dx)$$
$$+ u \sum_{m=2}^{\infty} z^m \sum_{k=0}^{m-1} u^k \int_{\overline{D}} e^{\lambda x}\mathbf{P}(T_m = k, S_m \in dx).$$

Using the fact that $\mathbf{P}(T_m = 0, S_m \in A) = 0$ for $A \subset D$ and $\mathbf{P}(T_m = m, S_m \in A) = 0$ for $A \subset \overline{D}$, we replace $\sum_{i=1}^{m}$ by $\sum_{i=0}^{m}$ and $\sum_{k=0}^{m-1}$ by $\sum_{k=0}^{m}$ on the right-hand side of the last equality. Thus,

$$zu\varphi(\lambda)(Q_0(z,u,\lambda) + Q(z,u,\lambda)) = Q_0(z,u,\lambda) - zu \int_D e^{\lambda x}\mathbf{P}(X_1 \in dx)$$
$$+ u\left(Q(z,u,\lambda) - z\int_{\overline{D}} e^{\lambda x}\mathbf{P}(X_1 \in dx)\right) = Q_0(z,u,\lambda) + uQ(z,u,\lambda) - zu\varphi(\lambda),$$

which coincides with the claim of the theorem. The theorem is proven.

From (1) it is immediate that

$$1 + \sum_{n=1}^{\infty} z^n \mathbf{E}(u^{T_n(D)} e^{\lambda S_n}) = 1 + Q(z,u,\lambda) + Q_0(z,u,\lambda)$$
$$= (1 - zu\varphi(\lambda))^{-1}(1 + (1-u)Q(z,u,\lambda)). \qquad (2)$$

Thus, to obtain the triple transform of the distribution of the pair (T_n, S_n) which is on the left-hand side of (2) it suffices to find $Q(z,u,\lambda)$.

Assume now that $D = [a,b]$ for some $a \le b$. Let

$$Q(z,u,\lambda) = Q_1(z,u,\lambda) + Q_2(z,u,\lambda),$$

where

$$Q_1(z,u,\lambda) = \sum_{n=1}^{\infty} z^n \sum_{k=0}^{n} u^k \int_{(-\infty,a)} e^{\lambda x} \mathbf{P}(T_n = k, S_n \in dx),$$

$$Q_2(z,u,\lambda) = \sum_{n=1}^{\infty} z^n \sum_{k=0}^{n} u^k \int_{(b,\infty)} e^{\lambda x} \mathbf{P}(T_n = k, S_n \in dx).$$

We turn to finding $Q_i(z,u,\lambda)$, $i = 1,2$.

Introduce the ladder epochs $\eta_\pm$ and the ladder heights $\chi_\pm$ as follows:

$$\eta_- = \inf\{n \ge 1 : S_n < 0\}, \quad \eta_+ = \inf\{n \ge 1 : S_n > 0\}, \quad \chi_\pm = S_{\eta_\pm}.$$

Here we put $\eta_+ = \infty$ if $S_n \le 0$ for all n and $\eta_- = \infty$ if $S_n \ge 0$ for all n. We do not define the quantities $\chi_\pm$ on the events $\{\eta_\pm = \infty\}$.

Let

$$R_\pm(z,\lambda) = 1 - \mathbf{E}(z^{\eta_\pm} \exp\{\lambda \chi_\pm\};\, \eta_\pm < \infty)$$

for $|z| \le 1$ and $\operatorname{Re}\lambda = 0$. These functions are components of the well-known factorization (for instance, see [7])

$$1 - z\varphi(\lambda) = R_+(z,\lambda) R_-(z,\lambda) R_0(z), \qquad (3)$$

where

$$R_0(z) = \exp\Big\{ -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathbf{P}(S_n = 0) \Big\}.$$

It is known that the following representations are valid for $|z| < 1$ and $\operatorname{Re}\lambda = 0$:

$$R_-(z,\lambda) = \exp\Big\{ -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathbf{E}(\exp\{\lambda S_n\}; S_n < 0) \Big\},$$

$$R_+(z,\lambda) = \exp\Big\{ -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathbf{E}(\exp\{\lambda S_n\}; S_n > 0) \Big\}.$$

Rearrange summands in (1) so that this identity takes the form

$$(1 + (1-u)Q(z,u,\lambda))(1 - z\varphi(\lambda)) = \left(1 + \frac{u-1}{u} Q_0(z,u,\lambda)\right)(1 - zu\varphi(\lambda))$$

and use the factorization of $1 - z\varphi(\lambda)$ and $1 - zu\varphi(\lambda)$, taking it into account that $R_\pm(z,\lambda) \ne 0$ for $|z| < 1$. We obtain

$$\frac{(1 + (1-u)(Q_1(z,u,\lambda) + Q_2(z,u,\lambda)))R_+(z,\lambda)}{R_+(zu,\lambda)} = \left(1 + \frac{u-1}{u} Q_0(z,u,\lambda)\right) \frac{R_-(zu,\lambda) R_0(zu)}{R_-(z,\lambda) R_0(z)}. \qquad (4)$$

From now on, we write $g(\lambda) \in S(A)$ if, for $\operatorname{Re}\lambda = 0$, the function g has the form

$$g(\lambda) = \int_A e^{\lambda x} dG(x), \quad \text{where} \int_A |dG(x)| < \infty.$$

Moreover, we use the notation

$$\left[\int_{-\infty}^{\infty} e^{\lambda x} dG(x)\right]^A = \int_A e^{\lambda x} dG(x).$$

For the functions of the variable λ in (2) we clearly have the relations

$$R_+^{\pm 1}(z,\lambda) \in S([0,\infty)), \quad R_-^{\pm 1}(z,\lambda) \in S((-\infty,0]),$$

$$1 + \frac{u-1}{u} Q_0(z,u,\lambda) \in S(\{0\} \cup D),$$

$$Q_1(z,u,\lambda) \in S((-\infty,a)), \quad Q_2(z,u,\lambda) \in S((b,\infty)).$$

For $a \le 0 \le b$ it thus follows that

$$\left[\left(1 + \frac{u-1}{u} Q_0(z,u,\lambda)\right) \frac{R_-(zu,\lambda) R_0(zu)}{R_-(z,\lambda) R_0(z)}\right]^{(b,\infty)} \equiv 0,$$

$$\left[\frac{Q_2(z,u,\lambda) R_+(z,\lambda)}{R_+(zu,\lambda)}\right]^{(b,\infty)} = \frac{Q_2(z,u,\lambda) R_+(z,\lambda)}{R_+(zu,\lambda)},$$

$$\left[\frac{R_+(z,\lambda)}{R_+(zu,\lambda)}\right]^{(b,\infty)} + (1-u)\left[\frac{Q_1(z,u,\lambda) R_+(z,\lambda)}{R_+(zu,\lambda)}\right]^{(b,\infty)} + (1-u)\frac{Q_2(z,u,\lambda) R_+(z,\lambda)}{R_+(zu,\lambda)} = 0,$$

from which we obtain

$$Q_2(z,u,\lambda) = \frac{R_+(zu,\lambda)}{(u-1)R_+(z,\lambda)}\left[\frac{R_+(z,\lambda)}{R_+(zu,\lambda)}\right]^{(b,\infty)} - \frac{R_+(zu,\lambda)}{R_+(z,\lambda)}\left[\frac{R_+(z,\lambda)}{R_+(zu,\lambda)} Q_1(z,u,\lambda)\right]^{(b,\infty)}. \tag{5}$$

For brevity, let $s = zu$. For each function g of the form

$$g(\lambda) = \int_{-\infty}^{\infty} e^{\lambda x}\, dG(x), \quad \text{where} \operatorname{Var} G = \int_{-\infty}^{\infty} |dG(x)| < \infty,$$

we define the operators $L_\pm$ as

$$(L_+ g)(z,s,\lambda) = \frac{R_+(s,\lambda)}{R_+(z,\lambda)}\left[\frac{R_+(z,\lambda)}{R_+(s,\lambda)} g(\lambda)\right]^{(b,\infty)},$$

$$(L_- g)(z,s,\lambda) = \frac{R_-(s,\lambda)}{R_-(z,\lambda)}\left[\frac{R_-(z,\lambda)}{R_-(s,\lambda)} g(\lambda)\right]^{(-\infty,a)}.$$

These operators depend themselves on z and s; however, for brevity, we do not show this dependence in the symbols for the operators. Alongside the variable λ, the function g can depend on z and s as well, and just this situation is most frequent in what follows.

Putting $h(\lambda) \equiv (u-1)^{-1}$, we rewrite (5) as

$$Q_2(z,u,\lambda) = (L_+ h)(z,s,\lambda) - (L_+ Q_1)(z,s,\lambda).$$

The symmetric arguments lead to the identity

$$Q_1(z,u,\lambda) = (L_- h)(z,s,\lambda) - (L_- Q_2)(z,s,\lambda)$$

from which we obtain

Theorem 2. *Let $D=[a,b]$, $a\le 0\le b$. Then, for $|z|<1$, $|uz|<1$, $\operatorname{Re}\lambda=0$, and $s=zu$, we have*

$$\begin{aligned} Q_1(z,u,\lambda) &= (L_-h)(z,s,\lambda)-(L_-L_+h)(z,s,\lambda)+(L_-L_+Q_1)(z,s,\lambda),\\ Q_2(z,u,\lambda) &= (L_+h)(z,s,\lambda)-(L_+L_-h)(z,s,\lambda)+(L_+L_-Q_2)(z,s,\lambda). \end{aligned} \tag{6}$$

Corollary 1. *If $D=[a,\infty)$ and $a\le 0$ then*

$$Q_1(z,u,\lambda)=(L_-h)(z,s,\lambda); \tag{7}$$

and, by analogy, if $D=(-\infty,b]$ and $b\ge 0$ then

$$Q_2(z,u,\lambda)=(L_+h)(z,s,\lambda). \tag{8}$$

Applying (2), we readily obtain

Corollary 2. *If $D=[a,\infty)$ and $a\le 0$ then*

$$1+\sum_{n=1}^{\infty} z^n \mathbf{E}(u^{T_n(D)}e^{\lambda S_n})=\frac{1+(1-u)(L_-h)(z,s,\lambda)}{1-s\varphi(\lambda)}. \tag{9}$$

If $D=(-\infty,b]$ and $b\ge 0$ then

$$1+\sum_{n=1}^{\infty} z^n \mathbf{E}(u^{T_n(D)}e^{\lambda S_n})=\frac{1+(1-u)(L_+h)(z,s,\lambda)}{1-s\varphi(\lambda)}. \tag{10}$$

Observe that

$$(u-1)(L_+h)(z,s,\lambda)=1-\frac{R_+(s,\lambda)}{R_+(z,\lambda)}\left[\frac{R_+(z,\lambda)}{R_+(s,\lambda)}\right]^{[0,b]}$$

and, hence, for $D=(-\infty,b]$,

$$1+\sum_{n=1}^{\infty} z^n \mathbf{E}(u^{T_n(D)}e^{\lambda S_n})=\frac{R_+(s,\lambda)}{(1-s\varphi(\lambda))R_+(z,\lambda)}\left[\frac{R_+(z,\lambda)}{R_+(s,\lambda)}\right]^{[0,b]}.$$

It is clear that, in the general case, finding the last factor on the right-hand side of this equality presents some difficulties. However, the calculation of L_+ is essentially simplified if $b=0$. Indeed, in this particular case, we have

$$\left[\frac{R_+(z,\lambda)}{R_+(s,\lambda)}\right]^{\{0\}}=1,$$

and so for $D=(-\infty,0]$ we obtain

$$1+\sum_{n=1}^{\infty} z^n \mathbf{E}(u^{T_n(D)}e^{\lambda S_n})=\frac{R_+(s,\lambda)}{(1-s\varphi(\lambda))R_+(z,\lambda)}.$$

On the other hand,

$$\sum_{n=1}^{\infty} z^n \mathbf{E}(u^{T_n(D)}e^{\lambda S_n})=\sum_{n=1}^{\infty} z^n \mathbf{E}(u^{n-T_n(\overline{D})}e^{\lambda S_n})=\sum_{n=1}^{\infty} s^n \mathbf{E}(v^{T_n(\overline{D})}e^{\lambda S_n}),$$

where $v=u^{-1}$; i.e., for $\overline{D}=(0,\infty)$ we have

$$1+\sum_{n=1}^{\infty} s^n \mathbf{E}(v^{T_n(\overline{D})}e^{\lambda S_n})=\frac{R_+(s,\lambda)}{(1-s\varphi(\lambda))R_+(sv,\lambda)}=\frac{1}{R_0(s)R_-(s,\lambda)R_+(sv,\lambda)}.$$

This result can be found in [1].

Calculating the expressions of the form $(L_\pm g)(z,s,\lambda)$ for the arbitrary values of a and b is rather simple in the case when the corresponding factorization component is a linear-fractional function. We state some needed assertions.

Lemma 1. *Let g be a function of the form*

$$g(\lambda) = \int_c^\infty e^{\lambda t} dG(t), \quad \operatorname{Re}\lambda = 0, \quad \operatorname{Var} G < \infty, \quad c \ge a.$$

Then

$$\left[\frac{g(\lambda)}{\lambda - \beta}\right]^{(-\infty,a)} = g(\beta)\frac{e^{(\lambda-\beta)a}}{\lambda - \beta}$$

for every $\beta < 0$.

Proof. We have

$$\left[\frac{g(\lambda)}{\lambda - \beta}\right]^{(-\infty,a)} = \left[\int_{-\infty}^{\infty} e^{\lambda t}\left(\int_{\max(c,t)}^{\infty} e^{-\beta(t-y)}\, dG(y)\right) dt\right]^{(-\infty,a)}$$

$$= \left[\int_{-\infty}^{\infty} e^{(\lambda-\beta)t}\, dt \int_{\max(c,t)}^{\infty} e^{\beta y} dG(y)\right]^{(-\infty,a)} = g(\beta)\int_{-\infty}^{a} e^{(\lambda-\beta)t}\, dt.$$

The following assertion is settled analogously.

Lemma 2. *Let g have the form*

$$g(\lambda) = \int_{-\infty}^c e^{\lambda t} dG(t), \quad \operatorname{Re}\lambda = 0, \quad \operatorname{Var} G < \infty, \quad c \le b.$$

Then

$$\left[\frac{g(\lambda)}{\lambda - \beta}\right]^{(b,\infty)} = g(\beta)\frac{e^{(\lambda-\beta)b}}{\lambda - \beta}$$

for every $\beta > 0$.

Suppose that

$$\mathbf{P}(X_1 < t) = ce^{\alpha t}, \quad t < 0. \tag{11}$$

In this case

$$\varphi(\lambda) = \frac{c}{\lambda + \alpha} + \int_0^\infty e^{\lambda t} d\mathbf{P}(X_1 < t).$$

It is easy to see that, for $|z| < 1$ and sufficiently large R, the inequality $|z\varphi(\lambda)| < 1$ is valid for all values of λ that lie on the contour

$$\Gamma = \{\lambda : |\lambda| = R, \operatorname{Re}\lambda < 0\} \cup \{\lambda : \operatorname{Re}\lambda = 0, |\lambda| \le R\}.$$

This means that the argument of $1 - z\varphi(\lambda)$ has no increment as λ traverses Γ. Therefore, by the well-known argument principle, this function has the same number of poles and zeros inside Γ for R large enough. Since there is a unique pole at $\lambda = -\alpha$, there exists a sole zero which we denote by $\lambda_-(z)$. We can now write

$$1 - z\varphi(\lambda) = \frac{(1 - z\varphi(\lambda))(\lambda + \alpha)}{\lambda - \lambda_-(z)}\frac{\lambda - \lambda_-(z)}{\lambda + \alpha}.$$

As a function of λ, the function $(1 - z\varphi(\lambda))(\lambda + \alpha)(\lambda - \lambda_-(z))^{-1}$ is analytic on the half-plane $\operatorname{Re}\lambda < 0$, continuous, bounded, and does not vanish in the closure of this set. The function $(\lambda - \lambda_-(z))(\lambda + \alpha)^{-1}$ has similar properties on the right half-plane. In view of the uniqueness properties for the factorization (3), we can put

$$R_+(z,\lambda) = C\frac{(1-z\varphi(\lambda))(\lambda+\alpha)}{\lambda-\lambda_-(z)}, \quad R_-(z,\lambda) = C^{-1}\frac{\lambda-\lambda_-(z)}{\lambda+\alpha}.$$

From (3) it follows that $\lim_{\lambda\to\infty} R_-(z,\lambda) = 1$; therefore, $C = C(z) \equiv 1$, although we could not specify the value of C, since this constant is cancelled in calculating expressions of the form $(L_\pm g)(z,s,\lambda)$.

So, by Lemma 1, for $a \le 0$ and every function g of the form

$$g(\lambda) = \int_0^\infty e^{\lambda t} dG(t), \quad \operatorname{Re}\lambda = 0, \quad \operatorname{Var} G < \infty,$$

we have

$$(L_-g)(z,s,\lambda) = \frac{\lambda-\lambda_-(s)}{\lambda-\lambda_-(z)}\left[\frac{\lambda-\lambda_-(z)}{\lambda-\lambda_-(s)}g(\lambda)\right]^{(-\infty,a)}$$

$$= \frac{(\lambda-\lambda_-(s))(\lambda_-(s)-\lambda_-(z))}{\lambda-\lambda_-(z)}\left[\frac{g(\lambda)}{\lambda-\lambda_-(s)}\right]^{(-\infty,a)} = \frac{(\lambda_-(s)-\lambda_-(z))}{\lambda-\lambda_-(z)}g(\lambda_-(s))e^{(\lambda-\lambda_-(s))a}.$$

We arrive now at the following:

Corollary 3. *Let $D = [a,\infty)$, $a \le 0$, and let (11) be satisfied. Then*

$$Q_1(z,u,\lambda) = \frac{(\lambda_-(s)-\lambda_-(z))}{(u-1)(\lambda-\lambda_-(z))}\; e^{(\lambda-\lambda_-(s))a}.$$

Using symmetric arguments, we can find $Q_2(z,u,\lambda)$ in the case of $D = (-\infty,b]$, $b \ge 0$, and $\mathbf{P}(X_1 > t) = ce^{-\alpha t}$, $t > 0$. Proceeding along the same lines with obvious replacement of integrals by sums, we can obtain similar results for the integer-valued random walks with $\mathbf{P}(X_1 = -k) = cp^{k-1}$, $k \ge 1$, or $\mathbf{P}(X_1 = k) = cp^{k-1}$, $k \ge 1$.

Return to the case of $D = [a,b]$, $a \le 0 \le b$.

Assume that the distribution X_1 has density of the form

$$f(t) = \begin{cases} c_1e^{-\alpha_1 t}, & t > 0, \\ c_2e^{\alpha_2 t}, & t \le 0, \end{cases} \tag{12}$$

where $\alpha_i > 0$, $c_1\alpha_2 + c_2\alpha_1 = \alpha_1\alpha_2$. In this case

$$\varphi(\lambda) = \frac{\lambda(c_2-c_1)-\alpha_1\alpha_2}{(\lambda-\alpha_1)(\lambda+\alpha_2)}.$$

The integral determining this function converges in the strip $-\alpha_2 < \operatorname{Re}\lambda < \alpha_1$, and

$$1 - z\varphi(\lambda) = \frac{\lambda^2-\lambda(\alpha_1-\alpha_2+z(c_2-c_1))+\alpha_1\alpha_2(z-1)}{(\lambda-\alpha_1)(\lambda+\alpha_2)} = \frac{(\lambda-\lambda_+(z))(\lambda-\lambda_-(z))}{(\lambda-\alpha_1)(\lambda+\alpha_2)}.$$

In view of the already-mentioned uniqueness properties for the factorization (3), we can put

$$R_+(z,\lambda) = \frac{\lambda-\lambda_+(z)}{\lambda-\alpha_1}, \quad R_-(z,\lambda) = \frac{\lambda-\lambda_-(z)}{\lambda+\alpha_2}.$$

Calculate $Q_2(z,u,\lambda)$. Using Lemmas 1 and 2, we find

$$(L_+h)(z,s,\lambda) = \frac{\lambda-\lambda_+(s)}{(u-1)(\lambda-\lambda_+(z))}\left[\frac{\lambda-\lambda_+(z)}{\lambda-\lambda_+(s)}\right]^{(b,\infty)}$$

$$= \frac{\lambda_+(s)-\lambda_+(z)}{(u-1)(\lambda-\lambda_+(z))}e^{(\lambda-\lambda_+(s))b}, \tag{13}$$

$$(L_-h)(z,s,\lambda) = \frac{\lambda_-(s)-\lambda_-(z)}{(u-1)(\lambda-\lambda_-(z))}e^{(\lambda-\lambda_-(s))a}, \tag{14}$$

$$(L_-Q_2)(z,s,\lambda) = \frac{\lambda_-(s)-\lambda_-(z)}{\lambda-\lambda_-(z)}Q_2(z,u,\lambda_-(s))e^{(\lambda-\lambda_-(s))a}, \tag{15}$$

$$(L_+L_-h)(z,s,\lambda) = \frac{\lambda_+(s)-\lambda_+(z)}{\lambda-\lambda_+(z)}(L_-h)(z,s,\lambda_+(s))e^{(\lambda-\lambda_+(s))b}$$
$$= \frac{\lambda_+(s)-\lambda_+(z)}{(u-1)(\lambda-\lambda_+(z))}H_1(z,s)\mu^{-a}(z,s)e^{(\lambda-\lambda_+(s))b}, \tag{16}$$

$$(L_+L_-Q_2)(z,s,\lambda) = \frac{\lambda_+(s)-\lambda_+(z)}{\lambda-\lambda_+(z)}(L_-Q_2)(z,s,\lambda_+(s))e^{(\lambda-\lambda_+(s))b}$$
$$= \frac{\lambda_+(s)-\lambda_+(z)}{\lambda-\lambda_+(z)}H_1(z,s)\mu^{-a}(z,s)Q_2(z,u,\lambda_-(s))e^{(\lambda-\lambda_+(s))b}. \tag{17}$$

Here and in the sequel, we use the notation

$$H_1(z,s) = \frac{\lambda_-(s)-\lambda_-(z)}{\lambda_+(s)-\lambda_-(z)}, \quad H_2(z,s) = \frac{\lambda_+(s)-\lambda_+(z)}{\lambda_-(s)-\lambda_+(z)},$$
$$H(z,s) = H_1(z,s)H_2(z,s), \quad \mu(s) = e^{\lambda_-(s)-\lambda_+(s)}.$$

Inserting the resultant expressions (13)–(17) into the second identity of (6) yields

$$Q_2(z,u,\lambda) = \frac{\lambda_+(s)-\lambda_+(z)}{(u-1)(\lambda-\lambda_+(z))}e^{(\lambda-\lambda_+(s))b}$$
$$\times(1-H_1(z,s)\mu^{-a}(z,s)+(u-1)H_1(z,s)\mu^{-a}(z,s)Q_2(z,u,\lambda_-(s))). \tag{18}$$

To find $Q_2(z,u,\lambda_-(s))$, we insert the value of $\lambda=\lambda_-(s)$ into (18). The so-obtained equation implies

$$(u-1)Q_2(z,u,\lambda_-(s)) = H_2(z,s)\mu^b(z,s)\frac{1-H_1(z,s)\mu^{-a}(s)}{1-H(z,s)\mu^{b-a}(s)}.$$

Inserting this expression into (18) and using symmetric arguments for $Q_1(z,u,\lambda)$, we arrive at the following result:

Theorem 3. *Let $D=[a,b]$, $a\le 0\le b$, and let* (12) *be satisfied. Then, for $|z|<1$, $|uz|<1$, $\operatorname{Re}\lambda=0$, and $s=zu$,*

$$Q_1(z,u,\lambda) = \frac{\lambda_-(s)-\lambda_-(z)}{(u-1)(\lambda-\lambda_-(z))}\frac{1-H_2(z,s)\mu^b(s)}{1-H(z,s)\mu^{b-a}(s)}e^{(\lambda-\lambda_-(s))a}, \tag{19}$$

$$Q_2(z,u,\lambda) = \frac{\lambda_+(s)-\lambda_+(z)}{(u-1)(\lambda-\lambda_+(z))}\frac{1-H_1(z,s)\mu^{-a}(s)}{1-H(z,s)\mu^{b-a}(s)}e^{(\lambda-\lambda_+(s))b}. \tag{20}$$

These expressions can be easily inverted in the variable λ. For instance, from (20) we obtain

Corollary 4. *Under the conditions of Theorem* 3, *for all $x\ge 0$ we have*

$$\sum_{n=1}^{\infty} z^n \sum_{k=0}^{n} u^k \mathbf{P}(T_n=k, S_n\ge b+x)$$
$$= \frac{(\lambda_+(s)-\lambda_+(z))(1-H_1(z,s)\mu^{-a}(s))}{(1-u)\lambda_+(z)(1-H(z,s)\mu^{b-a}(s))}e^{-\lambda_+(s)b-\lambda_+(z)x}.$$

Employing (2), we obtain

Corollary 5. *Under the conditions of Theorem* 3

$$1+\sum_{n=1}^{\infty} z^n \mathbf{E}(u^{T_n(D)}e^{\lambda S_n}) = \frac{1}{1-s\varphi(\lambda)}\left\{1-\frac{\lambda_-(s)-\lambda_-(z)}{\lambda-\lambda_-(z)}\,\frac{1-H_2(z,s)\mu^b(s)}{1-H(z,s)\mu^{b-a}(s)}e^{(\lambda-\lambda_-(s))a}\right.$$
$$\left.-\frac{\lambda_+(s)-\lambda_+(z)}{\lambda-\lambda_+(z)}\,\frac{1-H_1(z,s)\mu^{-a}(s)}{1-H(z,s)\mu^{b-a}(s)}e^{(\lambda-\lambda_+(s))b}\right\}.$$

Inverting the resultant representations in the variables z and u is a rather difficult problem and requires separate considerations.

The above-proposed method of finding $Q_i(z,u,\lambda)$ leads to explicit expressions in the more general case as well when $\varphi(\lambda)$ is a rational function. In this case factorization components are rational functions too (see [7]) and, hence, can be decomposed into partial fractions. As we can see, the use of the operators $L_\pm$ with partial fractions inside brackets is not difficult but the so-obtained expressions will be bulkier.

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