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FACTORIZATION REPRESENTATIONS IN THE BOUNDARY CROSSING PROBLEMS FOR RANDOM WALKS ON A MARKOV CHAIN

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Abstract: Let τ be some stopping time for a random walk S_n defined on transitions of a finite Markov chain and let $\tau(t)$ be the first passage time across the level t which occurs after τ. We prove a theorem that establishes a connection between the dual Laplace–Stieltjes transforms of the joint distributions of (τ, S_τ) and $(\tau(t), S_{\tau(t)})$. This result applies to the study of the number of crossings of a strip by sample paths of a random walk.

Keywords: Markov-modulated random walk, factorization representations, boundary crossing problems

1. Introduction

Let $\{\varkappa_n\}_{n\geq 0}$ be a finite homogeneous irreducible Markov chain with the state set $D = \{1, \dots, m\}$ and the matrix of transition probabilities $P = \|p_{jk}\|_{j,k\in D}$. Denote by $\{\xi_{jk}^{(n)}\}$, $n \geq 1$, $j, k \in D$, the family of independent random variables independent of $\{\varkappa_n\}$ and identically distributed for the fixed j and k.

The evolution of the Markov process $\{S_n, \varkappa_n\}_{n\geq 0}$ is determined by the initial value $\{0, \varkappa_0\}$ and relation

$$S_{n+1} = S_n + \xi_{\varkappa_n \varkappa_{n+1}}^{(n+1)}, \quad n \geq 0. \tag{1}$$

The distribution of $\{S_n, \varkappa_n\}_{n\geq 0}$ is completely defined by the distribution of $\varkappa_0$ and the matrix

$$F(\mu) = \left\| \int_{-\infty}^{\infty} e^{i\mu y} d\mathbb{P}(S_1 < y, \varkappa_1 = k/\varkappa_0 = j) \right\| = \|p_{jk} f_{jk}(\mu)\|, \quad \operatorname{Im}\mu = 0,$$

where $f_{jk}(\mu) = \mathbb{E}e^{i\mu\xi_{jk}^{(n)}}$.

Similar random walks are considered in many papers (for instance, see [1–4] and the references therein). It turned out that many results in the boundary crossing problems for ordinary random walks and the factorization methods connected with them can be translated to the Markov-modulated random walks. In this article, we prove Theorem 1 which is the translation of an analogous result of [1] to the processes (1). In [1] this result was established in the case $m = 1$ corresponding to the usual random walks generated by a sequence of independent identically distributed random variables.

Let τ be a stopping time for $\{S_n\}$. On the set $\{\tau < \infty\}$ we define the random variables

$$\tau_+(t) = \min\{n \geq \tau : S_n \geq t\}, \quad \tau_-(t) = \min\{n \geq \tau : S_n \leq t\},$$

where t is an arbitrary real number. In Theorem 1, we establish a connection between matrices of the dual Laplace–Stieltjes transforms

$$G_t^{\pm}(z, \mu) = \|\mathbb{E}(z^{\tau_\pm(t)} \exp\{i\mu S_{\tau_\pm(t)}\}; \tau_\pm(t) < \infty,\ \varkappa_{\tau_\pm(t)} = k/\varkappa_0 = j)\|$$

and

$$G(z, \mu) = \|\mathbb{E}(z^\tau e^{i\mu S_\tau}; \tau < \infty,\ \varkappa_\tau = k/\varkappa_0 = j)\|.$$

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We state the theorem in Section 2. Before stating it, we introduce the matrix factorization and give some new notations. Theorem 1 can be used in many boundary crossing problems for the random walk (1). We consider two of them below. The first relates to finding the dual Laplace–Stieltjes transforms of the joint distribution of the first exit time from a strip for the sample paths of (1) and of the location of a walking particle at that time. The second problem is to study the distribution of the number of crossings of a strip by the trajectories of (1). Using Theorem 1, we obtain the factorization representations for these two problems (see Section 3). We observe that, in many cases, further asymptotic analysis of the factorization representations of dual transforms leads to the complete asymptotic expansions of the distributions under consideration. It was carried out in [4] for the distributions related to the first exit time of (1) from a strip. Asymptotic expansions for the number of crossings of a strip which are based on Theorem 3 will be published later.

2. The Main Theorem

It is known [1] that, for $|z| < 1$ and $\mathrm{Im}\ \mu = 0$, the matrix function $E - zF(\mu)$ admits the canonical factorization of the following two types:

$$E - zF(\mu) = R_{z-}(\mu)R_{z+}(\mu) = L_{z+}(\mu)L_{z-}(\mu)$$

called the right and left factorizations. Here E is the unit matrix. We use the following representations for $R_{z+}(\mu)$ and $L_{z-}(\mu)$ which are known from [1]:

$$\begin{aligned} R_{z+}(\mu) &= E - \|\mathbb{E}(z^{\eta_+}e^{i\mu\chi_+}; \eta_+ < \infty, \varkappa_{\eta_+} = k/\varkappa_0 = j)\| \equiv E - \left\|R^{(1)}_{jk}(z,\mu)\right\|, \\ L_{z-}(\mu) &= E - \|\mathbb{E}(z^{\eta_-}e^{i\mu\chi_-}; \eta_- < \infty, \varkappa_{\eta_-} = k/\varkappa_0 = j)\| \equiv E - \left\|L^{(1)}_{jk}(z,\mu)\right\|. \end{aligned}$$

Here

$$\eta_+ = \inf\{n > 0 : S_n > 0\}, \quad \eta_- = \inf\{n > 0 : S_n \le 0\}, \quad \chi_\pm = S_{\eta_\pm}.$$

We assume everywhere that $\inf \varnothing = \infty$. Let h be the matrix function whose every element admits the representation

$$h_{j,k}(\mu) = \int_{-\infty}^{\infty} e^{i\mu y}\, dH_{j,k}(y), \quad \int_{-\infty}^{\infty} |dH_{j,k}(y)| < \infty \tag{2}$$

for $\mathrm{Im}\ \mu = 0$. We write this as

$$h(\mu) = \int_{-\infty}^{\infty} e^{i\mu y}\, dH(y)$$

where $H(y) = \|H_{j,k}(y)\|$. Following [2], define the operators

$$(\mathscr{A}_t h)(z,\mu) = \left[h(\mu)L^{-1}_{z-}(\mu)\right]^{(-\infty,t]} L_{z-}(\mu), \quad (\mathscr{B}_t h)(z,\mu) = \left[h(\mu)R^{-1}_{z+}(\mu)\right]^{[t,\infty)} R_{z+}(\mu)$$

with the notation

$$\left[\int_{-\infty}^{\infty} e^{i\mu y}\, dH(y)\right]^{D} = \int_{D} e^{i\mu y}\, dH(y), \quad D \subset R$$

(here we bear in mind the componentwise equality). The function h in the definition of the operators can depend on z as well.

Theorem 1. *For* $|z| < 1$ *and* $\operatorname{Im}\mu = 0$,

$$G_t^-(z,\mu) = (\mathscr{A}_t G)(z,\mu), \quad G_t^+(z,\mu) = (\mathscr{B}_t G)(z,\mu).$$

Proof. We use the following expressions for $R_{z+}^{-1}(\mu)$ and $L_{z-}^{-1}(\mu)$ known from [1]:

$$R_{z+}^{-1}(\mu) = E + \sum_{n=1}^{\infty} z^n \Big\| \int_0^{\infty} e^{i\mu y} d\mathbb{P}(\max_{0<m<n} S_m < S_n \le y, \varkappa_n = k/\varkappa_0 = j) \Big\|$$
$$\equiv E + \big\| R_{jk}^{(2)}(z,\mu) \big\|,$$

$$L_{z-}^{-1}(\mu) = E + \sum_{n=1}^{\infty} z^n \Big\| \int_{-\infty}^{0} e^{i\mu y} d\mathbb{P}(\min_{0<m<n} S_m \ge S_n, S_n \le y, \varkappa_n = k/\varkappa_0 = j) \Big\|$$
$$\equiv E + \big\| L_{jk}^{(2)}(z,\mu) \big\|.$$

Then

$$\mathbb{E}(z^{\tau_+(t)} e^{i\mu S_{\tau_+(t)}}; \tau_+(t) < \infty, \varkappa_{\tau_+(t)} = k/\varkappa_0 = j)$$
$$= \sum_{n=0}^{\infty} z^n \int_t^{\infty} e^{i\mu y} \Bigg(\mathbb{P}(\tau = n, S_\tau \in dy, \varkappa_n = k/\varkappa_0 = j)$$
$$+ \sum_{i_1=1}^{\infty} z^{i_1} \mathbb{P}(\tau = n, \sup_{n \le m < n+i_1} S_m < t, S_{\tau+i_1} \in dy, \varkappa_{n+i_1} = k/\varkappa_0 = j) \Bigg)$$
$$= \sum_{n=0}^{\infty} z^n \int_t^{\infty} e^{i\mu y} \mathbb{P}(\tau = n, S_\tau \in dy, \varkappa_n = k/\varkappa_0 = j)$$
$$+ \sum_{n=0}^{\infty} \sum_{i_1=1}^{\infty} z^{n+i_1} \int_t^{\infty} e^{i\mu y} \mathbb{P}(\tau = n, \sup_{n \le m < n+i_1} S_m < S_{n+i_1}, S_{n+i_1} \in dy, \varkappa_{n+i_1} = k/\varkappa_0 = j)$$
$$- \sum_{n=0}^{\infty} \sum_{i_1=1}^{\infty} z^{n+i_1} \int_t^{\infty} e^{i\mu y} \mathbb{P}(\tau = n, t \le \sup_{n \le m < n+i_1} S_m < S_{n+i_1},$$
$$S_{n+i_1} \in dy, \varkappa_{n+i_1} = k/\varkappa_0 = j) \equiv T_1 + T_2 - T_3.$$

Observe that

$$T_1 = \sum_{n=0}^{\infty} z^n \int_t^{\infty} e^{i\mu y} \mathbb{P}(\tau = n, S_\tau \in dy, \varkappa_n = k/\varkappa_0 = j) = [G_{jk}(z,\mu)]^{[t,\infty)}.$$

Moreover,

$$T_2 = \sum_{n=0}^{\infty} \sum_{i_1=1}^{\infty} z^{n+i_1} \int_t^{\infty} e^{i\mu y} \mathbb{P}(\tau = n, \sup_{n \le m < n+i_1} S_m < S_{n+i_1}, S_{n+i_1} \in dy, \varkappa_{n+i_1} = k/\varkappa_0 = j)$$
$$= \sum_{l=1}^{m} \sum_{n=0}^{\infty} \sum_{i_1=1}^{\infty} z^{n+i_1} \int_t^{\infty} e^{i\mu y} \mathbb{P}(\tau = n, \sup_{n \le m < n+i_1} S_m < S_{n+i_1}, S_{n+i_1} \in dy, \varkappa_n = l,$$

$$\varkappa_{n+i_1} = k/\varkappa_0 = j) = \Bigg[\sum_{l=1}^{m}\Bigg(\sum_{n=0}^{\infty} z^n \int\limits_{-\infty}^{\infty} e^{i\mu y}\mathbb{P}(\tau = n, S_\tau \in dy, \varkappa_n = l/\varkappa_0 = j)\Bigg)$$

$$\times\Bigg(\sum_{i_1=1}^{\infty} z^{i_1}\int\limits_{0}^{\infty} e^{i\mu y}\mathbb{P}(\sup_{0\le m<i_1} S_m < S_{i_1}, S_{i_1}\in dy, \varkappa_{i_1} = k/\varkappa_0 = l)\Bigg)\Bigg]^{[t,\infty)}$$

$$= \Bigg[\sum_{l=1}^{m} G_{jl}(z,\mu)R^{(2)}_{lk}(z,\mu)\Bigg]^{[t,\infty)}.$$

To analyze T_3, we introduce the following random variables on the set $\{\tau < \infty\}$:

$$\eta_1 = \inf\{n > 0 : S_{\tau+n} > S_\tau\}, \quad \chi_1 = S_{\tau+\eta_1} - S_\tau,$$

$$\eta_i = \inf\{n > 0 : S_{\tau+\eta_1+\dots+\eta_{i-1}+n} > S_{\tau+\eta_1+\dots+\eta_{i-1}}\},$$

$$\chi_i = S_{\tau+\eta_1+\dots+\eta_i} - S_{\tau+\eta_1+\dots+\eta_{i-1}}, \quad i \ge 2.$$

We obtain

$$T_3 = \sum_{n=0}^{\infty}\sum_{i_1=1}^{\infty} z^{n+i_1}\int\limits_{t}^{\infty} e^{i\mu y}\mathbb{P}(\tau = n, t \le \sup_{n\le m<n+i_1} S_m < S_{n+i_1}, S_{n+i_1}\in dy, \varkappa_{n+i_1} = k/\varkappa_0 = j)$$

$$= \sum_{n=0}^{\infty} z^n \int\limits_{t}^{\infty} e^{i\mu y}\Bigg[\sum_{i_1=1}^{\infty} z^{i_1}\mathbb{P}(\tau = n, S_\tau \ge t, S_\tau + \chi_1 \in dy, \eta_1 = i_1, \varkappa_{n+i_1} = k/\varkappa_0 = j)$$

$$+ \sum_{i_1,i_2=1}^{\infty} z^{i_1+i_2}\mathbb{P}(\tau = n, \eta_1 = i_1, S_\tau + \chi_1 \ge t,$$

$$\eta_2 = i_2, S_\tau + \chi_1 + \chi_2 \in dy, \varkappa_{n+i_1+i_2} = k/\varkappa_0 = j) + \dots\Bigg]$$

$$= \sum_{l=1}^{m}\sum_{n=0}^{\infty} z^n \int\limits_{t}^{\infty} e^{i\mu y}\Bigg[\sum_{i_1=1}^{\infty} z^{i_1}\int\limits_{t}^{\infty}\mathbb{P}(\tau = n, S_\tau \in du, \varkappa_n = l, S_\tau + \chi_1 \in dy, \eta_1 = i_1,$$

$$\varkappa_{n+i_1} = k/\varkappa_0 = j) + \sum_{i_1,i_2=1}^{\infty} z^{i_1+i_2}\int\limits_{t}^{\infty}\mathbb{P}(\tau = n, \eta_1 = i_1, S_\tau + \chi_1 \in du, \eta_2 = i_2,$$

$$S_\tau + \chi_1 + \chi_2 \in dy, \varkappa_{n+i_1} = l, \varkappa_{n+i_1+i_2} = k/\varkappa_0 = j) + \dots\Bigg].$$

Using the above notation

$$R^{(1)}_{lk}(z,\mu) = \sum_{i_1=1}^{\infty} z^{i_1}\int\limits_{0}^{\infty} e^{i\mu y}\mathbb{P}(\eta_1 = i_1, \chi_1 \in dy, \varkappa_{i_1} = k/\varkappa_0 = l),$$

we have

$$T_3 = \sum_{l=1}^{m}\Bigg([G_{jl}(z,\mu)]^{[t,\infty)} + \sum_{s=1}^{m}\sum_{n=0}^{\infty}\sum_{i_1=1}^{\infty} z^{n+i_1}\int\limits_{t}^{\infty} e^{i\mu y}\mathbb{P}(\tau = n,$$

$$\sup_{n\le m<n+i_1} < S_{n+i_1}, S_{n+i_1}\in dy, \varkappa_n = s, \varkappa_{n+i_1} = l/\varkappa_0 = j)\Bigg)R^{(1)}_{lk}(z,\mu)$$

$$
\begin{aligned}
&= \sum_{l=1}^{m} \Bigg([G_{jl}(z,\mu)]^{[t,\infty)} + \sum_{s=1}^{m} \Bigg(\sum_{n=0}^{\infty} z^{n} \int\limits_{t}^{\infty} e^{i\mu y} \mathbb{P}(\tau = n, S_\tau \in dy, \varkappa_n = s, /\varkappa_0 = j) \Bigg) \\
&\quad \times \Bigg(\sum_{i_1=1}^{\infty} z^{i_1} \int\limits_{0}^{\infty} e^{i\mu y} \mathbb{P}\big(\sup_{0 \le m < i_1} S_m < S_{i_1}, S_{i_1} \in dy, \varkappa_{i_1} = l/\varkappa_0 = s \big) \Bigg) R^{(1)}_{lk}(z,\mu) \Bigg) \\
&= \sum_{l=1}^{m} \big[G(z,\mu) R^{-1}_{z+}(\mu) \big]^{[t,\infty)}_{jl} R^{(1)}_{lk}(z,\mu).
\end{aligned}
$$

Summing the resultant expressions, we obtain one of the assertions of the theorem. The other assertion can be proven by analogy. The theorem is proven.

3. Factorization Representations of the Moment Generating Functions

1. Start with considering the problem of the first exit time from an interval (or from a strip if we consider paths on the plane) for the random walk (1). Let $a > 0$ and $b > 0$ be arbitrary numbers. Introduce the random variable

$$N = N(a,b) = \min\{n \ge 1 : S_n \notin (-a,b)\}$$

which is equal to the first exit time from the strip. Put

$$T_{jk}(z,A) = \sum_{n=1}^{\infty} z^{n} \mathbb{P}(S_N \in A, N = n, \varkappa_n = k/\varkappa_0 = j), \quad A \subset (-\infty,-a] \cup [b,\infty),$$

$$Q_1(z,\mu) = \Bigg\| \int\limits_{-\infty}^{-a+0} e^{i\mu y} T_{jk}(z,dy) \Bigg\|, \quad Q_2(z,\mu) = \Bigg\| \int\limits_{b}^{\infty} e^{i\mu y} T_{jk}(z,dy) \Bigg\|.$$

For brevity, we put $\mathscr{A}_{-a} = \mathscr{A}$ and $\mathscr{B}_b = \mathscr{B}$.

Theorem 2. *For* $|z| < 1$ *and* $\operatorname{Im}\mu = 0$,

$$Q_1(z,\mu) = (\mathscr{A}E)(z,\mu) - (\mathscr{A}Q_2)(z,\mu), \quad Q_2(z,\mu) = (\mathscr{B}E)(z,\mu) - (\mathscr{B}Q_1)(z,\mu).$$

These relations were proven in [4] by the Wiener–Hopf method. We show that they follow easily from Theorem 1. For instance, consider the second assertion of the theorem. The summand $(\mathscr{B}E)(z,\mu)$ (the dual transform of the joint distribution of the first passage time across the level b and of the location of a walking particle at that time) corresponds to all trajectories starting from zero and hitting the level b for the first time. Among these trajectories, we distinguish the trajectories that have the first exit from a strip through the upper boundary (to them there corresponds $Q_2(z,\mu)$) and trajectories that first leave a strip through the lower boundary and then hit the level b (to them there corresponds the summand $(\mathscr{B}Q_1)(z,\mu)$).

Inserting one of the resultant relations into the other, we arrive at the identities

$$
\begin{aligned}
Q_1(z,\mu) &= (\mathscr{A}E)(z,\mu) - (\mathscr{A}\mathscr{B}E)(z,\mu) + (\mathscr{A}\mathscr{B}Q_1)(z,\mu), \\
Q_2(z,\mu) &= (\mathscr{B}E)(z,\mu) - (\mathscr{B}\mathscr{A}E)(z,\mu) + (\mathscr{B}\mathscr{A}Q_2)(z,\mu).
\end{aligned}
$$

We can use these relations as the basis for a recurrent process, which actually is realization of the well-known principle "inclusion–exclusion" [6]. Recall that these relations gave grounds for obtaining the asymptotic expansions of the probabilities in [4].

2. Given arbitrary $a > 0$ and $b > 0$, we consider the random variables $\eta_n^{(1)}$ and $\eta_n^{(2)}$ respectively equal to the number of crossings of the strip $-a \le y \le b$ on the coordinate plane of points (x, y) from below to above and from above to below by a sample path of the random walk $\{(n, S_n)\}_{n=0}^{\infty}$.

Using Theorem 1, we obtain the factorization representations for the moment generating functions of the probabilities $\mathbb{P}\big(\eta_n^{(1)} = k/\varkappa_0 = s\big)$ and $\mathbb{P}\big(\eta_n^{(2)} = k/\varkappa_0 = s\big)$. Put

$$Q_i(z,k,s) = \sum_{n=1}^{\infty} z^n \mathbb{P}\big(\eta_n^{(i)} = k/\varkappa_0 = s\big), \quad i = 1, 2.$$

For the random walk under consideration, we define the stopping times $\tau_0^+ = \tau_0^- = 0$, $\tau_i^- = \inf\{n \ge \tau_{i-1}^+ : S_n \le -a\}$, $\tau_i^+ = \inf\{n \ge \tau_i^- : S_n \ge b\}$, $i \ge 1$.

From Theorem 1 follows that

$$\big\|\mathbb{E}(z^{\tau_k^+} \exp\{i\mu S_{\tau_k^+}\}; \tau_k^+ < \infty, \varkappa_{\tau_k^+} = l/\varkappa_0 = s)\big\| = ((\mathscr{B}\mathscr{A})^k E)(z,\mu) \tag{3}$$

for $|z| < 1$ and $\operatorname{Im}\mu = 0$. Consider another sequence of stopping times: $\nu_0^+ = \nu_0^- = 0$, $\nu_i^+ = \inf\{n \ge \nu_{i-1}^- : S_n \ge b\}$, $\nu_i^- = \inf\{n \ge \nu_i^+ : S_n \le -a\}$, $i \ge 1$. Then

$$\big\|\mathbb{E}(z^{\nu_k^-} \exp\{i\mu S_{\nu_k^-}\}; \nu_k^- < \infty, \varkappa_{\nu_k^-} = l/\varkappa_0 = s)\big\| = ((\mathscr{A}\mathscr{B})^k E)(z,\mu). \tag{4}$$

Putting $\mu = 0$ in (3) and (4), we obtain the moment generating functions of the probabilities $\mathbb{P}(\tau_k^+ = i, \varkappa_i = l/\varkappa_0 = s)$ and $\mathbb{P}(\nu_k^- = i, \varkappa_i = l/\varkappa_0 = s)$:

$$\left\|\sum_{i=1}^{\infty} z^i \mathbb{P}(\tau_k^+ = i, \varkappa_i = l/\varkappa_0 = s)\right\| = ((\mathscr{B}\mathscr{A})^k E)(z,0),$$

$$\left\|\sum_{i=1}^{\infty} z^i \mathbb{P}(\nu_k^- = i, \varkappa_i = l/\varkappa_0 = s)\right\| = ((\mathscr{A}\mathscr{B})^k E)(z,0).$$

Note that

$$\mathbb{P}\big(\eta_n^{(1)} \ge k, \varkappa_{\tau_k^+} = l/\varkappa_0 = s\big) = \sum_{i=1}^{n} \mathbb{P}\big(\tau_k^+ = i, \varkappa_i = l/\varkappa_0 = s\big),$$

$$\mathbb{P}\big(\eta_n^{(2)} \ge k, \varkappa_{\nu_k^-} = l/\varkappa_0 = s\big) = \sum_{i=1}^{n} \mathbb{P}(\nu_k^- = i, \varkappa_i = l/\varkappa_0 = s).$$

We now obtain the representation for $Q_1(z,k,s)$. We have

$$\left\|\sum_{n=1}^{\infty} z^n \mathbb{P}\big(\eta_n^{(1)} \ge k, \varkappa_{\tau_k^+} = l/\varkappa_0 = s\big)\right\| = \left\|\sum_{n=1}^{\infty} z^n \sum_{i=1}^{n} \mathbb{P}\big(\tau_k^+ = i, \varkappa_i = l/\varkappa_0 = s\big)\right\|$$
$$= (1 + z + z^2 + \dots)\left\|\sum_{i=1}^{\infty} z^i \mathbb{P}(\tau_k^+ = i, \varkappa_i = l/\varkappa_0 = s)\right\| = \frac{1}{1-z}((\mathscr{B}\mathscr{A})^k E)(z,0).$$

Analogously,

$$\left\|\sum_{n=1}^{\infty} z^n \mathbb{P}\big(\eta_n^{(1)} \ge k+1, \varkappa_{\tau_{k+1}^+} = l/\varkappa_0 = s\big)\right\| = \frac{1}{1-z}((\mathscr{B}\mathscr{A})^{k+1} E)(z,0).$$

Observe that

$$\begin{aligned}\mathbb{P}\big(\eta_n^{(1)} = k/\varkappa_0 = s\big) &= \sum_{l=1}^{m} \mathbb{P}\big(\eta_n^{(1)} \geq k, \varkappa_{\tau_k^+} = l/\varkappa_0 = s\big) \\ &\quad - \sum_{l=1}^{m} \mathbb{P}\big(\eta_n^{(1)} \geq k+1, \varkappa_{\tau_{k+1}^+} = l/\varkappa_0 = s\big)\end{aligned}$$

and, hence,

$$Q_1(z,k,s) = \sum_{l=1}^{m} \left\| \frac{1}{1-z} \{((\mathscr{B}\mathscr{A})^k E)(z,0) - ((\mathscr{B}\mathscr{A})^{k+1} E)(z,0)\} \right\|_{sl}.$$

The representation for $Q_2(z,k,s)$ can be obtained similarly. We have thus proven

Theorem 3. *For every $k \geq 0$ and $|z| < 1$,*

$$Q_1(z,k,s) = \sum_{l=1}^{m} \left\| \frac{1}{1-z} \{((\mathscr{B}\mathscr{A})^k E)(z,0) - ((\mathscr{B}\mathscr{A})^{k+1} E)(z,0)\} \right\|_{sl},$$

$$Q_2(z,k,s) = \sum_{l=1}^{m} \left\| \frac{1}{1-z} \{((\mathscr{A}\mathscr{B})^k E)(z,0) - ((\mathscr{A}\mathscr{B})^{k+1} E)(z,0)\} \right\|_{sl}.$$

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