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ASYMPTOTIC EXPANSIONS FOR THE DISTRIBUTION OF THE CROSSING NUMBER OF A STRIP BY A MARKOV-MODULATED RANDOM WALK

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Abstract: We obtain complete asymptotic expansions for the distribution of the crossing number of a strip in n steps by sample paths of a random walk defined on a finite Markov chain. We assume that the Cramér condition holds for the distribution of jumps and the width of the strip grows with n. The method consists in finding factorization representations of the moment generating functions of the distributions under study, isolating the main terms in the asymptotics of the representations, and inverting those main terms by the modified saddle-point method.

Keywords: Markov-modulated random walk, factorization representation, boundary crossing problem, asymptotic expansion

§ 1. Introduction

Consider a finite homogeneous irreducible Markov chain $\{\varkappa_n\}_{n\geq 0}$ with the set of states $D=\{1,\dots,m\}$, the matrix of transition probabilities $P = \|p_{jk}\|_{j,k\in D}$, and the stationary distribution $\pi = (\pi_1,\dots,\pi_m)$, $\pi_j > 0$, $j \in D$. Denote by $\{\xi_{jk}^{(n)}\}$, $n \geq 1$, $j,k \in D$, a family of independent random variables that are independent of $\{\varkappa_n\}$ and identically distributed for fixed j and k.

Introduce the Markov process $\{S_n,\varkappa_n\}_{n\geq 0}$ whose evolution is determined by the initial value $\{0,\varkappa_0\}$ and the relation $S_{n+1} = S_n + \xi^{(n+1)}_{\varkappa_n\varkappa_{n+1}}$, $n \geq 0$. The distribution of $\{S_n,\varkappa_n\}_{n\geq 0}$ will be completely defined by the distribution of $\varkappa_0$ and the matrix

$$F(\mu) = \left\| \int_{-\infty}^{\infty} e^{i\mu y} d\mathbb{P}(S_1 < y, \varkappa_1 = k/\varkappa_0 = j)\right\| = \|p_{jk}f_{jk}(\mu)\|, \tag{1}$$

where $f_{jk}(\mu) = \mathbb{E}e^{i\mu\xi^{(n)}_{jk}}$.

Given a random walk $\{S_n\}_{n=0}^{\infty}$, define the stopping times

$$\tau_0^+ = \tau_0^- = 0, \quad \tau_i^- = \inf\{n \geq \tau_{i-1}^+ : S_n \leq -a\},$$
$$\tau_i^+ = \inf\{n \geq \tau_i^- : S_n \geq b\}, \quad i \geq 1,\ a > 0,\ b > 0,$$

and consider the random variables η_1 and η_2 equal respectively to the numbers of upcrossings and downcrossings of the strip $-a \leq y \leq b$ on the coordinate plane of the points (x,y) by a sample path of the random walk $\{(n,S_n)\}_{n=0}^{\infty}$. It is obvious that the events $\{\eta_1 \geq k\}$ and $\{\tau_k^+ < \infty\}$ coincide. As follows from [1], the random variables η_i, $i = 1,2$, are finite with probability 1 for each initial state $\varkappa_0$ of the chain $\{\varkappa_n\}$ if, for instance, the “stationary” expectation of the random variable S_1,

$$\mathbb{E}_\pi S_1 = \sum_{j,k=1}^{m} \pi_j p_{jk}\mathbb{E}\xi^{(1)}_{jk},$$

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exists and differs from zero. For this case, in the article we find the asymptotics of the probabilities

$$\mathbb{P}(\eta_1 \geq k, \varkappa_{\tau_k^+} = l/\varkappa_0 = s)$$

as $a + b \to \infty$, which is the content of Theorem 2. Because of symmetry, we only consider the random variable η_1 here and in the sequel.

If $\mathbb{E}_\pi S_1 = 0$ then we introduce the random variable $\eta_n^{(1)}$ equal to the number of upward crossings of the strip by the path $\{(n, S_n)\}_{n=0}^\infty$ within the time span from 0 to n. Here $\eta_n^{(1)} = \max\{i \geq 0 : \tau_i^+ \leq n\}$. In Theorem 5 we obtain a complete asymptotic expansion of the probability

$$\mathbb{P}\big(\eta_n^{(1)} = k/\varkappa_0 = s\big)$$

in the powers of $1/\sqrt{n}$ as $n \to \infty$ assuming that the numbers a and b grow as $\sqrt{n}$, and the number k is fixed.

Our study is carried out by factorization technique in several stages, following the scheme proposed by A. A. Borovkov in [2] and subsequently implemented by many authors (see [3–5], for instance). The first stage involves constructing various kinds of factorization identities for the Fourier–Stieltjes transforms of the unknown distributions. In our case these are factorization representations of the matrix function

$$Q(z, \mu, k) \equiv \big\|\mathbb{E}\big(z^{\tau_k^+} \exp\{i\mu S_{\tau_k^+}\}; \tau_k^+ < \infty, \varkappa_{\tau_k^+} = l/\varkappa_0 = s\big)\big\|, \quad s, l = 1, \dots, m, \tag{2}$$

and the function

$$Q_1(z, k, s) \equiv \sum_{n=1}^{\infty} z^n \mathbb{P}\big(\eta_n^{(1)} = k/\varkappa_0 = s\big), \quad s = 1, \dots, m.$$

The results of this stage are not knew; they were published in [6], and we recall them in Section 3.

Putting $\mu = 0$ and $z = 1$ in (2), we obtain

$$\|\mathbb{P}(\eta_1 \geq k, \varkappa_{\tau_k^+} = l/\varkappa_0 = s)\| = \big\|\mathbb{P}\big(\tau_k^+ < \infty, \varkappa_{\tau_k^+} = l/\varkappa_0 = s\big)\big\| = Q(1, 0, k).$$

Thus, for $\mathbb{E}_\pi S_1 \neq 0$ this problem can be approached by studying the asymptotic behavior of $Q(z, 0, k)$ in a neighborhood of the point $z = 1$ as $a \to \infty$ and $b \to \infty$; after that we put $z = 1$. In order to determine the asymptotics of the probability $\mathbb{P}\big(\eta_n^{(1)} = k/\varkappa_0 = s\big)$ by contour integration, we also have to study the asymptotics of $Q_1(z, k, s)$ in a neighborhood of 1 as $a \to \infty$ and $b \to \infty$; after that we estimate this function outside the neighborhood using a suitable method. This constitutes the second stage of our study; we collect the so-obtained results in Theorems 1, 3, and 4.

At the final stage, while finding asymptotic expansions of the probability $\mathbb{P}\big(\eta_n^{(1)} = k/\varkappa_0 = s\big)$, we perform contour integration of the function $Q_1(z, k, s)$, show that the main contribution to the integral comes from the behavior of the function in a neighborhood of 1, and then use the asymptotic representation of $Q_1(z, k, s)$ of the previous stage. It includes the main part and a remainder. The remainder can be neglected; the asymptotics of the integral of the main part can be obtained by the modified saddle-point method of [2]. Implementing that, we use some technical tricks of [7]. Here we only consider the most important case of a and b growing as $\sqrt{n}$ although, as shown in [7], this method of asymptotic analysis enables us to obtain similar results in other situations compatible with the requirements $a \to \infty$, $b \to \infty$, and $a + b = o(n)$.

The coefficients in asymptotic expansions are given by a series of formulas, which requires much notation. Thus, we formulate the final result (Theorem 5) after the proof.

Henceforth we always assume fulfilled the following Cramér-type conditions:

I. There exists an entry of the matrix $F(\mu)$ such that the inverse Fourier image of its nth degree for some integer $n \geq 1$ includes an absolutely continuous component.

II. For all $j, k \in D$ the integrals in (1) converge absolutely for $v_+ \leq \operatorname{Im} \mu \leq v_-$; here $v_+ < 0$ and $v_- > 0$.

In [7] the authors obtained complete asymptotic expansions for the distribution of the crossing number of a strip by a sample path of an integer-valued random walk whose jumps are independent and identically distributed, which corresponds to the case $m = 1$. In this article we consider a more general construction ($m \geq 1$ is arbitrary); and in order to avoid repetitions, instead of the integrality of the random walk, we impose Condition I, given above, concerning the existence of an absolutely continuous component.

§ 2. Preliminaries

In this section we collect some useful results of [3, 4].

Denote by $V(\alpha,\beta)$ the Banach algebra of matrices of size m whose entries are the Fourier–Stieltjes transforms of left-continuous functions $\varphi(x)$ of bounded variation on each finite interval and such that $\int_{-\infty}^{\infty} e^{i\mu x}|d\varphi(x)|$ converges absolutely for $\alpha \le \operatorname{Im}\mu \le \beta$. Denote by $V_+(\alpha)$ the subset of $V(\alpha,\beta)$ consisting of the matrices of the form $\big\| \int_0^\infty e^{i\mu y} d\varphi_{jk}(y)\big\|$; denote by $V_-(\beta)$ the subset of $V(\alpha,\beta)$ consisting of the matrices of the form $\big\| \int_{-\infty}^{+0} e^{i\mu y} d\varphi_{jk}(y)\big\|$.

For $|z|<1$ and $v_+ \le \operatorname{Im}\mu \le v_-$ the matrix function $E - zF(\mu)$ (here E is the identity matrix) admits factorizations of the two types

$$E - zF(\mu) = R_{z-}(\mu)R_{z+}(\mu) = L_{z+}(\mu)L_{z-}(\mu), \tag{3}$$

called the *right* and *left* factorizations, respectively.

We mention some properties of the components of the factorization (3). The positive components $L_{z+}(\mu)$ and $R_{z+}(\mu)$, as functions of μ, belong to $V_+(v_+)$; the negative components $L_{z-}(\mu)$ and $R_{z-}(\mu)$ belong to $V_-(v_-)$. As functions of z taking values in the normed rings $V_\pm(v_\pm)$, the factorization components are analytic in the circle $|z|<1$; and, moreover, they have analytic continuations into a neighborhood of each point on the boundary of that circle for $\mathbb{E}_\pi S_1 \ne 0$. If $\mathbb{E}_\pi S_1 = 0$ then the factorization components are analytic functions of the variable $t = i\sqrt{z-1}$ in a neighborhood of $z=1$.

Suppose that each entry of some matrix function $f(\mu) = \|f_{jk}(\mu)\|$ admits, for $\operatorname{Im}\mu = 0$, some representation

$$f_{jk}(\mu) = \int_{-\infty}^{\infty} e^{i\mu y}\, d\widetilde{F}_{jk}(y), \qquad \int_{-\infty}^{\infty} |d\widetilde{F}_{jk}(y)| < \infty,$$

and $\widetilde{F}(y) = \|\widetilde{F}_{jk}(y)\|$. In this case, for brevity, we write

$$f(\mu) = \int_{-\infty}^{\infty} e^{i\mu y} d\widetilde{F}(y). \tag{4}$$

It is known [4] that the functions $R_{z\pm}^{\pm 1}(\mu)$ and $L_{z\pm}^{\pm 1}(\mu)$ admit representations of the form (4) for $|z|<1$ and $\operatorname{Im}\mu = 0$. Given some function f of the form (4), define the operators

$$(\mathscr{A} f)(z,\mu) = \big[f(\mu)L_{z-}^{-1}(\mu)\big]^{(-\infty,-a]} L_{z-}(\mu),$$

$$(\mathscr{B} f)(z,\mu) = \big[f(\mu)R_{z+}^{-1}(\mu)\big]^{[b,\infty)} R_{z+}(\mu),$$

where we use the notation

$$\Bigg[\int_{-\infty}^{\infty} e^{i\mu y}\, d\widetilde{F}(y)\Bigg]^D = \int_D e^{i\mu y} d\,\widetilde{F}(y), \quad D \subset R.$$

The function f can also depend on z.

Using these operators, in [7] we obtained the following factorization representations of the functions $Q(z,\mu,k)$ and $Q_1(z,k,s)$ which are applicable for $|z|<1$ and $\operatorname{Im}\mu = 0$:

$$Q(z,\mu,k) = \big\|\mathbb{E}\big(z^{\tau_k^+} \exp\{i\mu S_{\tau_k^+}\}; \tau_k^+ < \infty, \varkappa_{\tau_k^+} = l/\varkappa_0 = s\big)\big\| = ((\mathscr{B}\mathscr{A})^k E)(z,\mu),$$

$$Q_1(z,k,s) = \frac{1}{1-z}\sum_{l=1}^{m} \|\{((\mathscr{B}\mathscr{A})^k E)(z,0) - ((\mathscr{B}\mathscr{A})^{k+1} E)(z,0)\}\|_{sl}.$$

It is clear from these equalities that in order to obtain asymptotic representations of the functions $Q(z,0,k)$ and $Q_1(z,k,s)$ we have to find an asymptotic representation of the operator $((\mathscr{B}\mathscr{A})^k E)(z,0)$, which is carried out below.

Denote by $\lambda(\mu)$ the eigenvalue of $F(\mu)$ maximal in magnitude. Henceforth we assume that $\lambda(iv_+) > 1$ if $\mathbb{E}_\pi S_1 > 0$, and also that $\lambda(iv_-) > 1$ if $\mathbb{E}_\pi S_1 < 0$. If Condition II is fulfilled for some $\delta_1 > 0$ then the equation $1 - z\lambda(iv) = 0$ has at most two real solutions $v = \mu_\pm(z)$ with $\mu_+(z) \le 0 \le \mu_-(z)$ for $z \in [1-\delta_1, 1]$. The functions $\mu_+(z)$ and $\mu_-(z)$ admit analytic continuation into some δ-neighborhood of the interval $[1-\delta_1, 1]$ for $\mathbb{E}_\pi S_1 \ne 0$, and into the same neighborhood, but with a cut along the ray $z \ge 1$ in the case $\mathbb{E}_\pi S_1 = 0$.

Write $\mu_+ = \mu_+(1)$ and $\mu_- = \mu_-(1)$. For each sufficiently small $\delta > 0$ there exists $\gamma > 0$ such that if $|z-1| \ge \delta$, $|z| = 1$, then

$$R_{z\pm}^{-1}(\mu) \in V_\pm(\mu_\pm \mp \gamma), \quad L_{z\pm}^{-1}(\mu) \in V_\pm(\mu_\pm \mp \gamma).$$

Assume that $z \in L_\delta = \{z : |z| < 1,\ |z-1| < \delta\}$ and $\delta > 0$ is sufficiently small. Take a row eigenvector $I(\mu)$ and a column eigenvector $l(\mu)$ corresponding to the eigenvalue $\lambda(\mu)$ such that $l(\mu)I(\mu) = 1$. Following [3, 4], write

$$a_\pm = v_\pm \mp 1, \quad A_{z\pm} = I(i\mu_\pm(z))l(i\mu_\pm(z)),$$
$$F_{z\pm}(\mu) = E - (\mu_\pm(z) - a_\pm)(i\mu + \mu_\pm(z))^{-1} A_{z\pm}$$

and then

$$F_{z\pm}^{-1}(\mu) = E + (\mu_\pm(z) - a_\pm)(i\mu + a_\pm)^{-1} A_{z\pm},$$

and

$$\mathscr{R}_{z+}(\mu) \equiv R_{z+}(\mu)F_{z+}(\mu) \in V_+(v_+), \quad \mathscr{L}_{z+}(\mu) \equiv F_{z+}(\mu)L_{z+}(\mu) \in V_+(v_+),$$
$$\mathscr{R}_{z-}(\mu) \equiv F_{z-}(\mu)R_{z-}(\mu) \in V_-(v_-), \quad \mathscr{L}_{z-}(\mu) \equiv L_{z-}(\mu)F_{z-}(\mu) \in V_-(v_-).$$

Furthermore, for $z \in L_\delta$ there exists $\gamma > 0$ such that

$$\mathscr{R}_{z\pm}^{-1}(\mu) \in V_\pm(\mu_\pm \mp \gamma), \quad \mathscr{L}_{z\pm}^{-1}(\mu) \in V_\pm(\mu_\pm \mp \gamma).$$

§ 3. Asymptotic Representations for $Q(z,0,k)$ and $Q_1(z,k,s)$

As in [5], consider the functions

$$\mathscr{V}(z,\mu) \equiv -(i\mu + \mu_+(z))^{-1}(\mu_+(z) - a_+)A_{z+}\mathscr{R}_{z+}^{-1}(i\mu_+(z))R_{z+}(\mu),$$
$$\mathscr{U}(z,\mu) \equiv -(i\mu + \mu_-(z))^{-1}(\mu_-(z) - a_-)A_{z-}\mathscr{L}_{z-}^{-1}(i\mu_-(z))L_{z-}(\mu).$$

It is known that $\mathscr{V}(z,\mu) \in V_+(v_+)$ and $\mathscr{U}(z,\mu) \in V_-(v_-)$. Write

$$H_1(z) = \mathscr{U}(z, i\mu_+(z)), \quad H_2(z) = \mathscr{V}(z, i\mu_-(z)), \quad H(z) = H_2(z)H_1(z),$$
$$\mu(z) = e^{\mu_+(z) - \mu_-(z)}.$$

Below we write down inequalities between matrices and the operation of taking the absolute value of a matrix assuming that all those apply to each entry. Denote by the symbol C, possibly with indices, square $m \times m$ matrices with constant positive entries. The symbol c (with or without indices) always denotes positive constants, possibly distinct in different formulas.

Theorem 1. *There exist sufficiently small $\delta > 0$ and $\gamma > 0$ such that for $z \in L_\delta$ the representation*

$$Q(z,0,k) = \mu^{ka+(k-1)b}(z)H_1(z)H^{k-1}(z)e^{\mu_+(z)b}\mathscr{V}(z,\mu) + \Lambda(z,k) \tag{5}$$

is valid, where

$$|\Lambda(z,k)| \le e^{(\mu_+ - \mu_- - \gamma)(a+b)} M^{k-1}(z) C,$$
$$M(z) = C_0 |\mu^{(a+b)}(z)| \left(1 + \frac{e^{(\mu_+ - \gamma)(a+b)}}{e^{\operatorname{Re}\mu_+(z)(a+b)}}\right)\left(1 + \frac{e^{-(\mu_- + \gamma)(a+b)}}{e^{-\operatorname{Re}\mu_-(z)(a+b)}}\right).$$

We need a few lemmas to prove this theorem.

Lemma 1. *There exist $\delta > 0$ and $\gamma > 0$ such that for $z \in L_\delta$ the representations*

$$R_{z+}^{-1}(\mu) = -\frac{\mu_+(z) - a_+}{i\mu + \mu_+(z)} A_{z+} \mathscr{R}_{z+}^{-1}(i\mu_+(z)) + \psi_+(z,\mu),$$

$$L_{z-}^{-1}(\mu) = -\frac{\mu_-(z) - a_-}{i\mu + \mu_-(z)} A_{z-} \mathscr{L}_{z-}^{-1}(i\mu_-(z)) + \psi_-(z,\mu),$$

in which $\psi_\pm(z,\mu) \in V_\pm(\mu_\pm \mp \gamma)$, are valid.

Proof of Lemma 1. It is known that the functions $\mathscr{R}_{z+}(\mu)$ and $\mathscr{R}_{z-}(\mu)$ for some $\delta > 0$ and $\gamma > 0$ provide the canonical V-factorization (see Lemma 5.1 in [4]) of the function

$$F_{z-}(\mu)(E - zF(\mu))F_{z+}(\mu) = \mathscr{R}_{z-}(\mu)\mathscr{R}_{z+}(\mu)$$

for $z \in L_\delta$ and $\mu_+ - \gamma \le \operatorname{Im}\mu \le \mu_- + \gamma$. In addition, $\mathscr{R}_{z+}^{-1}(\mu) \in V_+(\mu_+ - \gamma)$ and $\mathscr{R}_{z-}^{-1}(\mu) \in V_-(\mu_- + \gamma)$. Further,

$$\begin{aligned} R_{z+}^{-1}(\mu) &= F_{z+}(\mu)\mathscr{R}_{z+}^{-1}(\mu) = (E - (\mu_+(z) - a_+)(i\mu + \mu_+(z))^{-1}A_{z+})\mathscr{R}_{z+}^{-1}(\mu) \\ &= \mathscr{R}_{z+}^{-1}(\mu) - \frac{\mu_+(z) - a_+}{i\mu + \mu_+(z)} A_{z+}\mathscr{R}_{z+}^{-1}(\mu) = \mathscr{R}_{z+}^{-1}(\mu) \\ &- \frac{\mu_+(z) - a_+}{i\mu + \mu_+(z)} A_{z+}\left[\mathscr{R}_{z+}^{-1}(i\mu_+(z)) + (i\mu + \mu_+(z))\frac{\mathscr{R}_{z+}^{-1}(\mu) - \mathscr{R}_{z+}^{-1}(i\mu_+(z))}{i\mu + \mu_+(z)}\right] \\ &\equiv -\frac{\mu_+(z) - a_+}{i\mu + \mu_+(z)} A_{z+}\mathscr{R}_{z+}^{-1}(i\mu_+(z)) + \psi_+(z,\mu). \end{aligned}$$

The function $\psi_+(z,\mu)$ lies in $V_+(\mu_+ - \gamma)$ by Proposition 5.2 in [4]. Similarly we obtain the claim for $L_{z-}^{-1}(\mu)$. Lemma 1 is proved.

Write

$$\psi_+(z,\mu) = \int_0^\infty e^{i\mu y}\, d\Psi_+(z,y), \quad \psi_-(z,\mu) = \int_{-\infty}^0 e^{i\mu y}\, d\Psi_-(z,y).$$

Lemma 2. 1. *There exists $\delta > 0$ such that for $z \in L_\delta$ and $\operatorname{Im}\mu \ge \mu_+$ for each function $g(\mu) = \int_{-\infty}^0 e^{i\mu y}\, dG(y) \in V_-(0)$ the representation*

$$(\mathscr{B}g)(z,\mu) = e^{(i\mu + \mu_+(z))b} g(i\mu_+(z))\mathscr{V}(z,\mu) + \Delta(z,\mu)R_{z+}(\mu)$$

is valid, where

$$\Delta(z,\mu) \equiv \int_b^\infty e^{i\mu y}\, d_y\varphi(z,y) = \int_b^\infty e^{i\mu y}\, d_y\Big(\int_{-\infty}^0 \Psi_+^T(z,y-t)dG^T(t)\Big)^T$$

(here the superscript T denotes transposition).

2. *If $g(z,\mu) = g(z,\mu)^{(-\infty;-a]} \in V_-(0)$ then*

$$\int_b^\infty |d_y\varphi(z,y)| \le \int_{-\infty}^{-a} |dG(y)| \int_{b+a}^\infty |d_y\Psi_+(z,y)|.$$

Lemma 3. 1. *There exists $\delta > 0$ such that for $z \in L_\delta$ and $\operatorname{Im}\mu \le \mu_-$ for each function $g(\mu) = \int_0^\infty e^{i\mu y} dG(y) \in V_+(0)$ the representation*

$$(\mathscr{A} g)(z,\mu) = e^{(i\mu+\mu_-(z))a} g(i\mu_-(z))\mathscr{U}(z,\mu) + \widetilde{\Delta}(z,\mu)L_{z-}(\mu)$$

is valid, where

$$\widetilde{\Delta}(z,\mu) \equiv \int_{-\infty}^{-a} e^{i\mu y}\, d_y\theta(z,y) = \int_{-\infty}^{-a} e^{i\mu y}\, d_y\Big(\int_0^\infty \Psi_-^T(z,y-t)\, dG^T(t)\Big)^T.$$

2. *If $g(\mu) = g(\mu)^{[b;\infty)} \in V_+(0)$ then*

$$\int_{-\infty}^{-a} |d_y\theta(z,y)| \le \int_b^\infty |dG(y)| \int_{-\infty}^{-(a+b)} |d_y\Psi_-(z,y)|.$$

Lemmas 2 and 3 were stated and proved in [5]. Here we only slightly modified their statements on using the notation we introduced. Note also that the statements of Lemmas 1–3 in the case $m = 1$ can be found in [8].

Proof of Theorem 1. Applying Lemmas 2 and 3 successively, we obtain the representation

$$\begin{aligned}((\mathscr{B}\mathscr{A})^k E)(z,\mu) = \Big\{&\mu^{ka+(k-1)b}(z)H_1(z)H^{k-1}(z)\\ &+\sum_{i=1}^{k}\mu^{(k-i)(a+b)}(z)\widetilde{\Delta}_i(z,i\mu_+(z))L_{z-}(i\mu_+(z))H^{k-i}(z)\\ &+\sum_{i=1}^{k-1}\mu^{(k-i)a+(k-i-1)b}(z)\Delta_i(z,i\mu_-(z))R_{z+}(i\mu_-(z))H_1(z)H^{k-i-1}(z)\Big\}\\ &\times e^{(i\mu+\mu_+(z))b}\mathscr{V}(z,\mu) + \Delta_k(z,\mu)R_{z+}(\mu).\end{aligned} \tag{6}$$

Put $\mu = 0$ in (6); then

$$((\mathscr{B}\mathscr{A})^k E)(z,0) = \mu^{ka+(k-1)b}(z)H_1(z)H^{k-1}(z)e^{\mu_+(z)b}\mathscr{V}(z,0) + \Lambda(z,k),$$

where

$$\begin{aligned}\Lambda(z,k) = \Big\{&\sum_{i=1}^{k}\mu^{(k-i)(a+b)}(z)\widetilde{\Delta}_i(z,i\mu_+(z))L_{z-}(i\mu_+(z))H^{k-i}(z)\\ &+\sum_{i=1}^{k-1}\mu^{(k-i)a+(k-i-1)b}(z)\Delta_i(z,i\mu_-(z))R_{z+}(i\mu_-(z))H_1(z)H^{k-i-1}(z)\Big\}\\ &\times e^{\mu_+(z)b}\mathscr{V}(z,0) + \Delta_k(z,0)R_{z+}(0).\end{aligned}$$

In order to estimate $|\Lambda(z,k)|$, we have to estimate $|\widetilde{\Delta}_i(z,i\mu_+(z)|$ and $|\Delta_i(z,i\mu_-(z))|$. To this end, we prove two lemmas. Write

$$(\mathscr{B}g)(z,\mu) \equiv \int_b^\infty e^{i\mu y} d_y G^B(z,y), \quad \mathscr{V}(z,\mu) \equiv \int_0^\infty e^{i\mu y} d_y v(z,y), \quad R_{z+}(\mu) \equiv \int_0^\infty e^{i\mu y} d_y r(z,y).$$

Henceforth, δ is always a sufficiently small positive number.

Lemma 4. *For each function $g(z,\mu)=\int_{-\infty}^{-a} e^{i\mu y}\, d_yG(z,y)\in V_-(0)$ the inequality*

$$\int\limits_b^\infty |d_yG^B(z,y)| \le \int\limits_{-\infty}^{-a} |d_yG(z,y)W_1(z)|$$

holds for $z\in L_\delta$, where

$$W_1(z)=C_1(e^{\operatorname{Re}\mu_+(z)(a+b)}+e^{(\mu_+-\gamma)(a+b)}).$$

Proof of Lemma 4. Start with finding an expression for $G^B(z,y)$. We have

$$(\mathscr{B}g)(z,\mu)=e^{(i\mu+\mu_+(z))b}g(i\mu_+(z))\mathscr{V}(z,\mu)+\Delta(z,\mu)R_{z+}(\mu)$$

$$=e^{(i\mu+\mu_+(z))b}g(i\mu_+(z))\int\limits_0^\infty e^{i\mu y}\, d_yv(z,y)+\int\limits_b^\infty e^{i\mu y}d_y\varphi(z,y)\int\limits_0^\infty e^{i\mu y}\, d_yr(z,y)$$

$$=e^{\mu_+(z)b}g(i\mu_+(z))\int\limits_b^\infty e^{i\mu y}\, d_yv(z,y-b)+\int\limits_b^\infty e^{i\mu y}\, d_y\int\limits_0^{y-b}\varphi(z,y-t)\, d_tr(z,t)\equiv\int\limits_b^\infty e^{i\mu y}\, d_yG^B(z,y).$$

Consequently,

$$G^B(z,y)=e^{\mu_+(z)b}g(i\mu_+(z))v(z,y-b)+\int\limits_0^{y-b}\varphi(z,y-t)\, d_tr(z,t),\quad y\ge b.$$

Furthermore,

$$\int\limits_b^\infty |d_yG^B(z,y)|\le |e^{\mu_+(z)b}g(i\mu_+(z))|\int\limits_b^\infty |d_yv(z,y-b)|+\int\limits_b^\infty \left|d_y\int\limits_0^{y-b}\varphi(z,y-t)\, d_tr(z,t)\right|$$

$$\le |e^{\mu_+(z)b}|\int\limits_{-\infty}^{-a}|e^{-\mu_+(z)y}||d_yG(z,y)|\int\limits_b^\infty |d_yv(z,y-b)|+\int\limits_b^\infty \left|d_y\int\limits_0^{y-b}\varphi(z,y-t)\, d_tr(z,t)\right|$$

$$\le |e^{\mu_+(z)(a+b)}|\int\limits_{-\infty}^{-a}|d_yG(z,y)|\int\limits_0^\infty |d_yv(z,y)|+\int\limits_b^\infty |d_y\varphi(z,y)|\int\limits_0^\infty |d_tr(z,t)|$$

$$\le |e^{\mu_+(z)(a+b)}|\int\limits_{-\infty}^{-a}|d_yG(z,y)|\int\limits_0^\infty |d_yv(z,y)|+e^{(\mu_+-\gamma)(a+b)}\int\limits_{-\infty}^{-a}|d_yG(z,y)|C\int\limits_0^\infty |d_tr(z,t)|$$

$$\le \int\limits_{-\infty}^{-a}|d_yG(z,y)|\Big(|e^{\mu_+(z)(a+b)}|\int\limits_0^\infty |d_yv(z,y)|+e^{(\mu_+-\gamma)(a+b)}C\int\limits_0^\infty |d_yr(z,y)|\Big)$$

$$\le \int\limits_{-\infty}^{-a}|d_yG(z,y)|C_1(e^{\operatorname{Re}\mu_+(z)(a+b)}+e^{(\mu_+-\gamma)(a+b)}).$$

Lemma 4 is proved.

Similarly, write

$$(\mathscr{A}g)(z,\mu)\equiv\int\limits_{-\infty}^{-a} e^{i\mu y}d_yG^A(z,y),\quad \mathscr{U}(z,\mu)\equiv\int\limits_{-\infty}^{0} e^{i\mu y}d_yu(z,y),\quad L_{z-}(\mu)\equiv\int\limits_{-\infty}^{0} e^{i\mu y}d_yl(z,y).$$

Lemma 5. *For each function $g(z,\mu)=\int_b^\infty e^{i\mu y}\,d_yG(z,y)\in V_+(0)$ the inequality*

$$\int\limits_{-\infty}^{-a}|d_yG^A(z,y)|\le\int\limits_b^\infty|d_yG(z,y)|W_2(z)$$

holds for $z\in L_\delta$, where

$$W_2(z)=C_2(e^{-\operatorname{Re}\mu_-(z)(a+b)}+e^{-(\mu_-+\gamma)(a+b)}).$$

Proof of Lemma 5 goes similarly to the proof of Lemma 4.

Return to proving Theorem 1. Applying Lemmas 4 and 5 in turn, we obtain

$$|\widetilde{\Delta}_i(z,i\mu_+(z))|\le\int\limits_{-\infty}^{-a}|e^{-\mu_+(z)y}||d_y\theta_i(z,y)|\le|e^{\mu_+(z)a}|\int\limits_{-\infty}^{-a}|d_y\theta_i(z,y)|$$

$$\le|e^{\mu_+(z)a}|\int\limits_b^\infty|d_yG^{(BA)^{i-1}}(z,y)|\int\limits_{-\infty}^{-(a+b)}|d_y\Psi_-(z,y)|$$

$$\le|e^{\mu_+(z)a}(W_1(z)W_2(z))^{i-1}|e^{-(\mu_-+\gamma)(a+b)}C_3.$$

Similarly,

$$|\Delta_i(z,i\mu_-(z))|\le|e^{-\mu_-(z)b}(W_2(z)W_1(z))^{i-1}W_2(z)|e^{(\mu_+-\gamma)(a+b)}C_4.$$

Write

$$C_5=\sup_{z\in L_\delta}\max(C_1,C_2,C_3,C_4,|H_1(z)|,|H_2(z)|,|R_{z+}(i\mu_-(z))|,|R_{z+}(0)|,|L_{z+}(i\mu_+(z))|,|\mathscr{V}(z,0)|).$$

Then

$$S_1\equiv\left|\sum_{i=1}^k\mu^{(k-i)(a+b)}(z)\widetilde{\Delta}_i(z,i\mu_+(z))L_{z-}(i\mu_+(z))H^{k-1}(z)\mathscr{V}(z,0)e^{\mu_+(z)b}\right|$$

$$\le e^{-(\mu_-+\gamma)(a+b)}|e^{\operatorname{Re}\mu_+(z)(a+b)}|\sum_{i=1}^k|\mu^{(k-i)(a+b)}(z)|(W_2(z)W_1(z))^{i-1}C_5^{2(k-i)+3}$$

$$\le e^{(\mu_+-\mu_--\gamma)(a+b)}\sum_{i=1}^k|\mu^{(k-i)(a+b)}(z)|(e^{\operatorname{Re}\mu_+(z)(a+b)}+e^{(\mu_+-\gamma)(a+b)})^{i-1}$$

$$\times(e^{-\operatorname{Re}\mu_-(z)(a+b)}+e^{-(\mu_-+\gamma)(a+b)})^{i-1}C_5^{2(i-1)}C_5^{2(k-i)+3}$$

$$=|\mu^{(k-1)(a+b)}(z)|e^{(\mu_+-\mu_--\gamma)(a+b)}C_5^{2(k-1)}C_5^3$$

$$\times\sum_{i=1}^k\left(1+\frac{e^{(\mu_+-\gamma)(a+b)}}{e^{\operatorname{Re}\mu_+(z)(a+b)}}\right)^{i-1}\left(1+\frac{e^{-(\mu_-+\gamma)(a+b)}}{e^{-\operatorname{Re}\mu_-(z)(a+b)}}\right)^{i-1}$$

$$\le e^{(\mu_+-\mu_--\gamma)(a+b)}\left|\mu^{(a+b)}(z)\left(1+\frac{e^{(\mu_+-\gamma)(a+b)}}{e^{\operatorname{Re}\mu_+(z)(a+b)}}\right)\left(1+\frac{e^{-(\mu_-+\gamma)(a+b)}}{e^{-\operatorname{Re}\mu_-(z)(a+b)}}\right)C_5^2\right|^{k-1}C_0$$

$$=e^{(\mu_+-\mu_--\gamma)(a+b)}M^{k-1}(z)C_0.$$

Similarly we obtain the inequalities

$$S_2\equiv\left|\sum_{i=1}^{k-1}\mu^{(k-i)a+(k-i-1)b}(z)\Delta_i(z,i\mu_-(z))R_{z+}(i\mu_-(z))\right.$$

$$\left.\times\widetilde{H}^{k-i-1}(z)H_1(z)e^{\mu_+(z)b}\mathscr{V}(z,0)\right|\le e^{(\mu_+-\mu_--\gamma)(a+b)}M^{k-1}(z)C_0,$$

$$|\Delta_k(z,0)R_{z+}(0)|\le e^{(\mu_+-\mu_--\gamma)(a+b)}M^{k-1}(z)C_0.$$

Adding up these inequalities, we arrive at the claim of Theorem 1.

Suppose now that

$$\mathbb{E}_\pi S_1 = \sum_{j,k=1}^{m} \pi_j p_{jk} \mathbb{E}\xi_{jk}^{(1)} < 0.$$

Then the equation $1 - \lambda(iv) = 0$ has two roots $\mu_+ = 0$ and $\mu_- \equiv h > 0$. Using the inequality

$$\|\mathbb{P}(\eta_1 \geq k, \varkappa_{\tau_k^+} = l/\varkappa_0 = s)\| = Q(1, 0, k)$$

and letting z go to 1 in (5), we obtain the following

Theorem 2. *If $\mathbb{E}_\pi S_1 < 0$ then for each $k \geq 1$*

$$\|\mathbb{P}(\eta_1 \geq k, \varkappa_{\tau_k^+} = l/\varkappa_0 = s)\| = e^{-h(ka+(k-1)b}H_1(1)H^{k-1}(1)\mathscr{V}(1,0)(E + \Theta(a,b,k)),$$

where $|\Theta(a,b,k)| \leq e^{-\gamma(a+b)}C_k$, $\gamma > 0$, and C_k is a matrix with constant nonnegative entries.

From now on we assume that the crossing number of the strip in question is infinite with probability 1. As was mentioned already, this is so provided that $\mathbb{E}_\pi S_1 = 0$. Recall that $\eta_n^{(1)}$ is the number of upcrossings of the strip $-a \leq y \leq b$ up to and including time n, and

$$Q_1(z,k,s) = \sum_{n=1}^{\infty} z^n \mathbb{P}\big(\eta_n^{(1)} = k/\varkappa_0 = s\big).$$

Below we use the following assertion of [5].

The vector $I(i\mu_+(z))$ is an eigenvector of the matrix $H(z)$ corresponding to the eigenvalue $h(z)$. The function $h(z)$ is analytic in a neighborhood of zero in the variable $t = i\sqrt{z-1}$, and $h(1) = 1$.

Theorem 3. *Suppose that $\mathbb{E}_\pi S_1 = 0$. Then there exist $\delta > 0$ and $\gamma > 0$ such that for $z \in L_\delta$ the representation*

$$Q_1(z,k,s) = \frac{1 - \mu^{a+b}h(z)}{1-z} \sum_{l=1}^{m} \Big\{ \|\mu^{ka+(k-1)b}(z)e^{\mu_+(z)b}h^{k-1}(z)H_1(z)\mathscr{V}(z,0)\|_{sl}$$
$$+ \frac{1}{1-z}\|\Lambda(z,k) - \Lambda(z,k+1)\|_{sl}\Big\}$$

is valid, where

$$|\Lambda(z,k)| \leq e^{-\gamma(a+b)}M^{k-1}(z)C_0,$$
$$M(z) = |\mu^{(a+b)}(z)| \left(1 + \frac{e^{-\gamma(a+b)}}{e^{\mu_+(z)(a+b)}}\right)\left(1 + \frac{e^{-\gamma(a+b)}}{e^{-\mu_-(z)(a+b)}}\right) C.$$

Proof of Theorem 3. It is known [3] that

$$Q_1(z,k,s) = \frac{1}{1-z}\sum_{l=1}^{m} \|\{((\mathscr{B}\mathscr{A})^k E)(z,0) - ((\mathscr{B}\mathscr{A})^{k+1}E)(z,0)\}\|_{sl}.$$

Consequently,

$$Q_1(z,k,s) = \frac{1}{1-z}\sum_{l=1}^{m} \|Q(z,0,k) - Q(z,0,k+1)\|_{sl}.$$

Theorem 1 yields

$$Q_1(z,k,s) = \frac{1}{1-z}\sum_{l=1}^{m}\Big\{\|\mu^{ka+(k-1)b}(z)e^{\mu_+(z)b}H_1(z)(1 - \mu^{a+b}H(z))$$
$$\times H^{k-1}(z)\mathscr{V}(z,0)\|_{sl} + \frac{1}{1-z}\|\Lambda(z,k) - \Lambda(z,k+1)\|_{sl}\Big\}.$$

It remains to use the definition of $\mathscr{V}(z,\mu)$ and the statement before Theorem 3. Theorem 3 is proved.

For subsequent contour integration of the moment generating function $Q_1(z,k,s)$ we need to estimate it on the set $l_1 = \{|z-1| \geq \delta, |z| = 1\}$.

Theorem 4. *Suppose that $\mathbb{E}_\pi S_1 = 0$ and $\delta > 0$ is arbitrary. Then there exist positive constants $\gamma > 0$ and $c_k > 0$ such that for all $s = 1, \dots, m$ and $z \in l_1$ we have $|Q_1(z,k,s)| \le c_k e^{-\gamma(a+b)k}$.*

Proof of Theorem 4. In order to estimate $Q_1(z,k,s)$ it suffices to estimate $((\mathscr{B}\mathscr{A})^k E)(z,0)$ on the set l_1. Write

$$f_k(z,\mu) \equiv ((\mathscr{B}\mathscr{A})^k E)(z,\mu) \equiv \int_b^\infty e^{i\mu y}\, d_y D_k(z,y)$$

and show that

$$\int_b^\infty |d_y D_k(z,y)| \le e^{-\gamma(a+b)k} C^k, \tag{7}$$

where C is a matrix of positive constants. Use induction; take $k=1$ and write

$$L_{z-}^{-1}(\mu) = \int_\infty^0 e^{i\mu y} d\bar{l}(z,y), \quad R_{z+}^{-1}(\mu) = \int_0^\infty e^{i\mu y} d\bar{r}(z,y).$$

Then

$$(\mathscr{A}E)(z,\mu) = \big[L_{z-}^{-1}(\mu)\big]^{(-\infty,-a]} L_{z-}(\mu) = \int_{-\infty}^{-a} e^{i\mu y}\, d_y \bar{l}(z,y) \int_{-\infty}^{0} e^{i\mu y}\, d_y l(z,y)$$
$$= \int_{-\infty}^{-a} e^{i\mu y}\, d_y \int_{y+a}^{0} \bar{l}(z,y-t)\, d_t l(z,t) \equiv \int_{-\infty}^{-a} e^{i\mu y}\, d_y D_{0A}(z,y).$$

Further,

$$\left[\int_{-\infty}^{-a} e^{i\mu y}\, d_y D_{0A}(z,y) R_{z+}^{-1}(\mu)\right]^{[b,\infty)} = \left[\left(\int_0^\infty e^{i\mu y}\, d_y \bar{r}^T(z,y) \int_{-\infty}^{-a} e^{i\mu y}\, d_y D_{0A}^T(z,y)\right)^T\right]^{[b,\infty)}$$
$$= \int_b^\infty e^{i\mu y}\, d_y \left(\int_{-\infty}^{-a} \bar{r}^T(z,y-t)\, d_t D_{0A}^T(z,t)\right)^T \equiv \int_b^\infty e^{i\mu y}\, d_y \overline{D}_{0BA}(z,y).$$

Therefore,

$$f_1(z,\mu) = \int_b^\infty e^{i\mu y} d_y \overline{D}_{0BA}(z,y) R_{z+}(\mu) = \int_b^\infty e^{i\mu y} d_y \overline{D}_{0BA}(z,y) \int_0^\infty e^{i\mu y} d_y r(z,y)$$
$$= \int_b^\infty e^{i\mu y} d_y \int_0^{y-b} \overline{D}_{0BA}(z,y-t) d_t r(z,t) \equiv \int_b^\infty e^{i\mu y} d_y D_1(z,y).$$

Estimate

$$\int_b^\infty |d_y D_1(z,y)| = \int_b^\infty \left| d_y \int_0^{y-b} \overline{D}_{0BA}(z,y-t) d_t r(z,t) \right| \le \int_b^\infty |d_y \overline{D}_{0BA}(z,y)| \int_0^\infty |d_t r(z,t)|.$$

On the other hand,

$$\int\limits_b^\infty |d_y\overline{D}_{0BA}(z,y)| = \int\limits_b^\infty \left| d_y(\int\limits_{-\infty}^{-a} \bar{r}^T(z,y-t)d_tD_{0A}^T(z,t))^T \right| \le \int\limits_{-\infty}^{-a} |d_tD_{0A}(z,t)| \int\limits_b^\infty |d_y\bar{r}(z,y)|,$$

$$\int\limits_{-\infty}^{-a} |d_yD_{0A}(z,y)| = \int\limits_{-\infty}^{-a} \left| d_y \int\limits_{y+a}^{0} \bar{l}(z,y-t)d_tl(z,t) \right| \le \int\limits_{-\infty}^{-a} |d_y\bar{l}(z,y)| \int\limits_{-\infty}^{0} |d_tl(z,t)|.$$

Therefore,

$$\int\limits_b^\infty |d_yD_1(z,y)| \le \int\limits_{-\infty}^{-a} |d_y\bar{l}(z,y)| \int\limits_{-\infty}^{0} |d_tl(z,t)| \int\limits_b^\infty |d_y\bar{r}(z,y)| \int\limits_b^\infty |d_yr(z,y)|.$$

Furthermore, $R_{z+}^{-1} \in V_+(-\gamma)$ and $L_{z-}^{-1} \in V_-(\gamma)$; consequently,

$$\int\limits_0^\infty e^{\gamma y}|d_y\bar{r}(z,y)| < \infty, \qquad \int\limits_{-\infty}^{0} e^{-\gamma y}|d_y\bar{l}(z,y)| < \infty.$$

Then

$$\int\limits_b^\infty |d_y\bar{r}(z,y)| = \int\limits_b^\infty e^{\gamma y-\gamma y}|d_y\bar{r}(z,y)| \le e^{-\gamma b}\int\limits_b^\infty e^{\gamma y}|d_y\bar{r}(z,y)|,$$

$$\int\limits_{-\infty}^{-a} |d_y\bar{l}(z,y)| \le \int\limits_{-\infty}^{-a} e^{\gamma y-\gamma y}|d_y\bar{l}(z,y)| \le e^{-\gamma a}\int\limits_{-\infty}^{-a} e^{-\gamma y}|d_y\bar{l}(z,y)|.$$

Thus,

$$\int\limits_b^\infty |d_yD_1(z,y)| \le e^{-\gamma(a+b)} \int\limits_{-\infty}^{-a} e^{-\gamma y}|d_y\bar{l}(z,y)| \int\limits_{-\infty}^{0} |d_tl(z,t)| \int\limits_b^\infty e^{\gamma y}|d_y\bar{r}(z,y)| \int\limits_0^\infty |d_yr(z,y)| \equiv e^{-\gamma(a+b)}C,$$

and so (7) is valid for $k=1$. Show that the validity of (7) for $k=\nu$ implies its validity for $k=\nu+1$. Find

$$(\mathscr{A} f_\nu)(z,\mu) = \left[f_\nu(z,\mu)L_{z-}^{-1}(\mu)\right]^{(-\infty,-a]}L_{z-}(\mu).$$

We have

$$[f_\nu(z,\mu)L_{z-}^{-1}(\mu)]^{(-\infty,-a]} = \left[\left(\int\limits_{-\infty}^{0} e^{i\mu y}d_y\bar{l}^T(z,y) \int\limits_b^\infty e^{i\mu y}d_yD_\nu^T(z,y) \right)^T\right]^{(-\infty,-a]}$$

$$= \int\limits_{-\infty}^{-a} e^{i\mu y}\, d_y\left(\int\limits_b^\infty \bar{l}^T(z,y-t)\, d_tD_\nu^T(z,t) \right)^T \equiv \int\limits_{-\infty}^{-a} e^{i\mu y}\, d_y\overline{D}_{\nu A}(z,y),$$

$$\left[f_\nu(z,\mu)L_{z-}^{-1}(\mu)\right]^{(-\infty,-a]}L_{z-}(\mu) = \int\limits_{-\infty}^{-a} e^{i\mu y}\, d_y\overline{D}_{\nu A}(z,y) \int\limits_{-\infty}^{0} e^{i\mu y}\, d_yl(z,y)$$

$$= \int\limits_{-\infty}^{-a} e^{i\mu y}\, d_y \int\limits_{y+a}^{0} \overline{D}_{\nu A}(z,y-t)\, d_tl(z,t) \equiv \int\limits_{-\infty}^{-a} e^{i\mu y}\, d_yD_{\nu A}(z,y).$$

Find $(\mathscr{B}\mathscr{A} f_\nu)(z,\mu)$. We have

$$\Big[\int_{-\infty}^{-a} e^{i\mu y}\, d_y D_{\nu A}(z,y) R_{z+}^{-1}(\mu)\Big]^{[b,\infty)} = \Big[\Big(\int_0^\infty e^{i\mu y} d_y \bar r^T(z,y) \int_{-\infty}^{-a} e^{i\mu y} d_y D_{\nu A}^T(z,y)\Big)^T\Big]^{[b,\infty)}$$

$$= \int_b^\infty e^{i\mu y} d_y \Big(\int_{-\infty}^{-a} \bar r^T(z,y-t) d_t D_{\nu A}^T(z,t)\Big)^T \equiv \int_b^\infty e^{i\mu y} d_y \overline{D}_{\nu BA}(z,y),$$

$$f_{\nu+1}(z,\mu) = \int_b^\infty e^{i\mu y}\, d_y \overline{D}_{\nu BA}(z,y) R_{z+}(\mu) = \int_b^\infty e^{i\mu y}\, d_y \overline{D}_{\nu BA}(z,y) \int_0^\infty e^{i\mu y}\, d_y r(z,y)$$

$$= \int_b^\infty e^{i\mu y}\, d_y \int_0^{y-b} \overline{D}_{\nu BA}(z,y-t)\, d_t r(z,t) \equiv \int_b^\infty e^{i\mu y}\, d_y D_{\nu+1}(z,y).$$

Estimate

$$\int_b^\infty |d_y D_{\nu+1}(z,y)| = \int_b^\infty \Big|d_y \int_0^{y-b} \overline{D}_{\nu BA}(z,y-t)\, d_t r(z,t)\Big| \le \int_b^\infty |d_y \overline{D}_{\nu BA}(z,y)| \int_0^\infty |d_t r(z,t)|.$$

On the other hand,

$$\int_b^\infty |d_y \overline{D}_{\nu BA}(z,y)| = \int_b^\infty \Big|d_y (\int_{-\infty}^{-a} \bar r^T(z,y-t) d_t D_{\nu A}^T(z,t))^T\Big| \le \int_{-\infty}^{-a} |d_t D_{\nu A}(z,t)| \int_b^\infty |d_y \bar r(z,y)|,$$

$$\int_{-\infty}^{-a} |d_y D_{\nu A}(z,y)| = \int_{-\infty}^{-a} \Big|d_y \int_{y+a}^{0} \overline{D}_{\nu A}(z,y-t) d_t l(z,t)\Big| \le \int_{-\infty}^{-a} |d_y \overline{D}_{\nu A}(z,y)| \int_{-\infty}^{0} |d_t l(z,t)|.$$

The last estimate is

$$\int_{-\infty}^{-a} |d_y \overline{D}_{\nu A}(z,y)| = \int_{-\infty}^{-a} \Big|d_y \Big(\int_b^\infty \bar l^T(z,y-t) d_t D_\nu^T(z,t)\Big)^T\Big| \le \int_b^\infty |d_t D_\nu(z,t)| \int_{-\infty}^{-a} |d_y \bar l(z,y)|.$$

Thus,

$$\int_b^\infty |d_y D_{\nu+1}(z,y)| \le e^{-\gamma(a+b)} \int_b^\infty |d_y D_\nu(z,y)| C.$$

The estimate (7) is established. It follows that

$$|f_k(z,0)| \le \int_b^\infty |d_y D_k(z,y)| \le e^{-\gamma(a+b)k} C^k.$$

Therefore,

$$|Q_1(z,k,s)| \le c_k e^{-\gamma(a+b)k}.$$

Theorem 4 is proved.

§ 4. Asymptotic Expansions of Probabilities

In this section we derive complete asymptotic expansions of the probabilities $\mathbb{P}\big(\eta_n^{(1)} = k/\varkappa_0 = s\big)$ as $n \to \infty$ for each fixed k while assuming that $a = x_1\sqrt{n}$ and $b = x_2\sqrt{n}$, where x_1 and x_2 are arbitrary fixed positive numbers.

Denote by l_2 the contour obtained from the arc $\{|z-1| < \delta, |z| = 1\}$ by bending it inside L_δ near the point $z = 1$. We have

$$\mathbb{P}\big(\eta_n^{(1)} = k/\varkappa_0 = s\big) = \frac{1}{2\pi i}\int\limits_{l_1} Q_1(z,k,s)z^{-n-1}\,dz + \frac{1}{2\pi i}\int\limits_{l_2} Q_1(z,k,s)z^{-n-1}\,dz. \tag{8}$$

By Theorem 4, the first integral in (8) admits the estimate

$$\frac{1}{2\pi i}\int\limits_{l_1} Q_1(z,k,s)z^{-n-1}\,dz = O(e^{-C\sqrt{n}}).$$

In the second integral in (8), along the contour l_2, replace $Q_1(z,k,s)$ with the asymptotic representation obtained in Theorem 3. Write

$$J(n,k,s) \equiv \frac{1}{2\pi i}\sum_{l=1}^{m}\left\|\int\limits_{l_2}\frac{1-\mu^{a+b}(z)h(z)}{(1-z)z^{n+1}}\mu^{ka+(k-1)b}(z)e^{\mu_+(z)b}h^{k-1}(z)H_1(z)\mathscr{V}(z,0)\,dz\right\|_{sl},$$

$$J_1(n,k,s) \equiv \frac{1}{2\pi i}\sum_{l=1}^{m}\left\|\int\limits_{l_2}\frac{\Lambda(z,k)-\Lambda(z,k+1)}{(1-z)z^{n+1}}\,dz\right\|_{sl};$$

then

$$\frac{1}{2\pi i}\int\limits_{l_2} Q_1(z,k,s)z^{-n-1}\,dz = J(n,k,s) + J_1(n,k,s).$$

Consider the integral $J_1(n,k,s)$. By the choice of the contour l_2 we have $|1-z| > \delta_2$ for $z \in l_2$ and some $\delta_2 > 0$. Theorem 3 yields the estimate

$$|\Lambda(z,k)| \le Ce^{-\gamma(x_1+x_2)\sqrt{n}};$$

hence,

$$|J_1(n,k,s)| \le ce^{-\gamma(x_1+x_2)\sqrt{n}}.$$

Note that the functions $\mu(z)$, $\mu_+(z)$, $h(z)$, $H_1(z)$, $\mathscr{V}(z,0)$ and $\Lambda(z,k)$ admit series expansions into the nonnegative powers of $i\sqrt{z-1}$ in some neighborhood of 1 cut along the ray $\{z = \operatorname{Re} z \ge 1\}$. This follows from their definitions and the articles [4, 5]. Moreover, $h(1) = \mu(1) = 1$. We further make the substitution $t = i(z-1)^{1/2}$ (where we choose the principal value of the root) and, for convenience, keep the same notation for the functions h, H_1, μ, μ_+, $\mathscr{V}$ and Λ. It is clear that all these functions (now as functions of t) are analytic in a neighborhood of zero. Denote by l_3 a contour obtained from the contour $\{|\arg(z-1)| = \pi/4, |z-1| \le \delta\}$ by bending inside K_δ near the point $z = 1$, where $K_\delta = \{|z-1| \le \delta, |\arg(z-1)| > \pi/4\}$. Denote by l_4 and l_5 the line segments that join the endpoints of l_1 and l_3 and lie respectively in the half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$.

Write

$$\Pi(t) = -\frac{1}{\pi i}\sum_{l=1}^{m}\left\|\frac{1-\mu^{a+b}(t)h(t)}{t(1-t^2)^{n+1}}\mu^{ka+(k-1)b}(t)e^{\mu_+(t)b}h^{k-1}(t)H_1(t)\mathscr{V}(t,0)\,dt\right\|_{sl}.$$

Let Γ, Γ_2, Γ_4, and Γ_5 be the images of the curves l_3, l_2, l_4, and l_5 in the plane of the variable t respectively. By the Cauchy Theorem and Theorem 4 for

$$d(n) \equiv \int\limits_{\Gamma_4 \cup \Gamma_5} \Pi(t)\,dt = \int\limits_{\Gamma_2} \Pi(t)\,dt - \int\limits_{\Gamma} \Pi(t)\,dt$$

we have the estimate

$$|d(n)| = O(c^k e^{-\gamma k (x_1+x_2)\sqrt{n}}).$$

Write

$$\widetilde{J}(n,k) \equiv \int\limits_{\Gamma} \Pi(t)\,dt.$$

Then

$$J(n,k) = \widetilde{J}(n,k) + O(c^k e^{-\gamma k (x_1+x_2)\sqrt{n}}).$$

Consider

$$\begin{aligned}
\widetilde{J}(n,k) &= -\frac{1}{\pi i}\sum_{l=1}^{m}\left\|\int\limits_{\Gamma}\frac{1-\mu^{a+b}(t)h(t)}{t(1-t^2)^{n+1}}\mu^{ka+(k-1)b}(t)e^{\mu_+(t)b}h^{k-1}(t)H_1(t)\mathscr{V}(t,0)\,dt\right\|_{sl}\\
&= -\frac{1}{\pi i}\sum_{l=1}^{m}\left\|\int\limits_{\Gamma}\frac{1-h(t)}{t(1-t^2)^{n+1}}\mu^{ka+(k-1)b}(t)e^{\mu_+(t)b}h^{k-1}(t)H_1(t)\mathscr{V}(t,0)\,dt\right\|_{sl}\\
&\quad -\frac{1}{\pi i}\sum_{l=1}^{m}\left\|\int\limits_{\Gamma}\frac{1-\mu^{a+b}(t)}{t(1-t^2)^{n+1}}\mu^{ka+(k-1)b}(t)e^{\mu_+(t)b}h^{k}(t)H_1(t)\mathscr{V}(t,0)\,dt\right\|_{sl}\\
&= \frac{1}{\pi i}\sum_{l=1}^{m}\left\|\int\limits_{\Gamma}\widetilde{D}(t)e^{n\tilde{f}(t)}\,dt\right\|_{sl} + \frac{a+b}{\pi i}\sum_{l=1}^{m}\left\|\int\limits_{0}^{1}\int\limits_{\Gamma}D(t)e^{nf(t)}\,dtdx\right\|_{sl}\\
&= I_1(n) + (a+b)I_2(n).
\end{aligned}$$

In order to obtain the last equality, we use the representation

$$\frac{e^{(\mu_+(t)-\mu_-(t))(a+b)}-1}{(\mu_+(t)-\mu_-(t))(a+b)} = \int\limits_0^1 e^{(\mu_+(t)-\mu_-(t))(a+b)x}\,dx,$$

change the order of integration in the double integral, and introduce the notation

$$\begin{gathered}
\widetilde{D}(t) = -\frac{1-h(t)}{(1-t^2)th(t)}H_1(t)\mathscr{V}(t,0), \quad D(t) = \frac{\mu_+(t)-\mu_-(t)}{(1-t^2)t}H_1(t)\mathscr{V}(t,0),\\
\tilde{f}(t) = \mu_+(t)(kx_1+kx_2)\tau_1 - \mu_-(t)(kx_1+(k-1)x_2)\tau_1 + k\tau_2\log h(t) - \log(1-t^2),\\
f(t) = \mu_+(t)((x_1+x_2)x+kx_1+kx_2)\tau_1 - \mu_-(t)((x_1+x_2)x+kx_1+(k-1)x_2)\tau_1\\
+k\tau_2\log h(t) - \log(1-t^2),\\
\tau_1 = 1/\sqrt{n}, \quad \tau_2 = 1/n.
\end{gathered}$$

Here we choose the principal value of the logarithm. Recall from [5] that in a neighborhood of zero we have the expansions

$$\mu_\pm(t) = \pm\psi_1 t + \psi_2 t^2 \pm \psi_3 t^3 + \dots, \quad \psi_1 > 0.$$

Consider the equations

$$F_1(t,z_1,z_2) \equiv [(\mu'_+(t)(kx_1+kx_2) - \mu'_-(t)(kx_1+(k-1)x_2)]z_1 + k\frac{h'(t)}{h(t)}z_2 + \frac{2t}{1-t^2} = 0, \tag{9}$$

$$F_2(t,z_1,z_2) \equiv [\mu'_+(t)((x_1+x_2)x + kx_1 + kx_2) - \mu'_-(t)((x_1+x_2)x + kx_1 + (k-1)x_2)]z_1 + k\frac{h'(t)}{h(t)}z_2 + \frac{2t}{1-t^2} = 0. \tag{10}$$

The functions $F_j(t,z_1,z_2)$, $j=1,2$, are analytic at the point $\mathbf{0}=(0,0,0)$, with $F_j(\mathbf{0})=0$ and $\frac{\partial F_j(\mathbf{0})}{\partial t}=2$; consequently, by the Implicit Function Theorem there exist solutions $t_1(z_1,z_2)$ and $t_2(z_1,z_2)$ to (9) and (10) respectively representable in some neighborhood $\Delta=\{|z_j|<\tau, j=1,2\}$ by converging double power series in z_1 and z_2. Note that $\tilde{f}'(t)=F_1(t,\tau_1,\tau_2)$ and $f'(t)=F_2(t,\tau_1,\tau_2)$. Write

$$h(t)=1+h_1t+h_2t^2+\dots,\quad \log h(t)=\eta_1 t+\eta_2 t^2+\dots,\quad \log(1-t^2)=-t^2-\frac{t^4}{2}-\frac{t^6}{3}-\dots.$$

Recall that $\tau_1=1/\sqrt{n}$ and $\tau_2=1/n$. Then the expansions for the saddle points $\tilde{t}_0$ and t_0 of the functions $\tilde{f}(t)$ and $f(t)$ respectively have the form

$$\tilde{t}_0 = -\psi_1\left(kx_1+\frac{2k-1}{2}x_2\right)\frac{1}{\sqrt{n}} - \left(\frac{kh_1}{2} - 2\psi_1\psi_3x_2\left(kx_1+\frac{2k-1}{2}x_2\right)\right)\frac{1}{n} + O\left(\frac{1}{(\sqrt{n})^3}\right),$$

$$t_0 = -\psi_1\left((x_1+x_2)x+kx_1+\frac{2k-1}{2}x_2\right)\frac{1}{\sqrt{n}} - \left(\frac{kh_1}{2} - 2\psi_1\psi_3x_2\left((x_1+x_2)x+kx_1+\frac{2k-1}{2}x_2\right)\right)\frac{1}{n} + O\left(\frac{1}{(\sqrt{n})^3}\right).$$

Note that

$$\tilde{f}(t) = \left(\psi_1(2kx_1+(2k-1)x_2)\frac{1}{\sqrt{n}} + k\eta_1\frac{1}{n}\right)t + \left(\psi_2x_2\frac{1}{\sqrt{n}} + k\eta_2\frac{1}{n}+1\right)t^2 + \left(\psi_3(2kx_1+(2k-1)x_2)\frac{1}{\sqrt{n}} + k\eta_3\frac{1}{n}\right)t^3 + \dots,$$

$$f(t) = \left(\psi_1(2(x_1+x_2)x+2kx_1+(2k-1)x_2)\frac{1}{\sqrt{n}} + k\eta_1\frac{1}{n}\right)t + \left(\psi_2x_2\frac{1}{\sqrt{n}} + k\eta_2\frac{1}{n}+1\right)t^2 + \left(\psi_3((x_1+x_2)x+2kx_1+(2k-1)x_2)\frac{1}{\sqrt{n}} + k\eta_3\frac{1}{n}\right)t^3 + \dots.$$

Then

$$\tilde{f}(\tilde{t}_0) = -\psi_1^2\left(kx_1+\frac{2k-1}{2}x_2\right)^2\frac{1}{n} - \psi_1\left(kx_1+\frac{2k-1}{2}x_2\right) \times\left(k\eta_1 - \psi_1\psi_3x_2\left(kx_1+\frac{2k-1}{2}x_2\right)\right)\frac{1}{(\sqrt{n})^3} + O\left(\frac{1}{n^2}\right),$$

$$f(t_0) = -\psi_1^2\left((x_1+x_2)x+kx_1+\frac{2k-1}{2}x_2\right)^2\frac{1}{n} - \psi_1\left((x_1+x_2)x+kx_1+\frac{2k-1}{2}x_2\right) \times\left(k\eta_1 - \psi_1\psi_3x_2\left((x_1+x_2)x+kx_1+\frac{2k-1}{2}x_2\right)\right)\frac{1}{(\sqrt{n})^3} + O\left(\frac{1}{n^2}\right).$$

Further,

$$\tilde f''(\tilde t_0) = 2 + 2\psi_2 x_2 \frac{1}{\sqrt{n}} + (2k\eta_2 - 3\psi_1\psi_3(2kx_1 + (2k-1)x_2)^2)\frac{1}{n} + O\left(\frac{1}{(\sqrt{n})^3}\right),$$

$$f''(t_0) = 2 + 2\psi_2 x_2 \frac{1}{\sqrt{n}} + (2k\eta_2 - 3\psi_1\psi_3(2(x_1+x_2)x + 2kx_1$$
$$+(2k-1)x_2)^2)\frac{1}{n} + O\left(\frac{1}{(\sqrt{n})^3}\right).$$

Apply to $I_1(n)$ and $I_2(n)$ the modified saddle-point method. Since $\tilde f'(\tilde t_0) = 0$, $f'(t_0) = 0$, each pair of the curves $\operatorname{Re}\tilde f(t) = \tilde f(\tilde t_0)$ and $\operatorname{Re} f(t) = f(t_0)$ splits the neighborhoods of the points $\tilde t_0$ and t_0 respectively into four rectangular sectors, in which in turn $\operatorname{Re}\tilde f(t) < \tilde f(\tilde t_0)$, $\operatorname{Re} f(t) < f(t_0)$ and $\operatorname{Re}\tilde f(t) > \tilde f(\tilde t_0)$, $\operatorname{Re} f(t) > f(t_0)$. For sufficiently large n the endpoints of the contour Γ lie inside the sectors $\operatorname{Re}\tilde f(t) < \tilde f(\tilde t_0)$, $\operatorname{Re} f(t) < f(t_0)$. In $I_1(n)$ and $I_2(n)$ modify the contour Γ so that, while remaining inside those sectors, it passes through the points $\tilde t_0$ and t_0.

Denote by $\tilde g_{l,s}$ and $g_{l,s}$ the coefficients of z^l in the products

$$\frac{1}{s!}(\widetilde D_0 + \widetilde D_1 z + \dots)(\tilde f_1 z + \tilde f_2 z^2 + \dots)^s, \quad \frac{1}{s!}(D_0 + D_1 z + \dots)(f_1 z + f_2 z^2 + \dots)^s$$

respectively, where

$$\widetilde D_m = \frac{\widetilde D^{(m)}(\tilde t_0)}{m!}, \quad \tilde f_m = \frac{\tilde f^{(m+2)}(\tilde t_0)}{(m+2)!},$$

$$D_m = \frac{D^{(m)}(t_0)}{m!}, \quad f_m = \frac{f^{(m+2)}(t_0)}{(m+2)!}, \quad m = 0, 1, \dots.$$

Then for each $q \geq 1$ the Laplace method for estimating integrals yields

$$I_1(n) = \sum_{l=0}^{m} \left\| \sum_{j=0}^{q-1} \frac{1}{n^{j+1/2}} \widetilde Q_j e^{n\tilde f(\tilde t_0)} + \frac{1}{n^{q+1/2}} \widetilde R_q e^{n\tilde f(\tilde t_0)} \right\|_{sl}, \tag{11}$$

$$I_2(n) = \sum_{l=0}^{m} \left\| \sum_{j=0}^{q-1} \frac{1}{n^{j+1/2}} \int_0^1 Q_j e^{nf(t_0)}\, dx + \frac{1}{n^{q+1/2}} \int_0^1 R_q e^{nf(t_0)}\, dx \right\|_{sl}, \tag{12}$$

$$\widetilde Q_j = \frac{(-1)^{1/2}}{\pi} \sum_{s=0}^{2j} \tilde g_{2j,s} (-\tilde f_0)^{-s-j-1/2} \Gamma(j+s+1/2), \tag{13}$$

$$Q_j = \frac{(-1)^{1/2}}{\pi} \sum_{s=0}^{2j} g_{2j,s} (-f_0)^{-s-j-1/2} \Gamma(j+s+1/2). \tag{14}$$

Under our assumptions, $e^{n\tilde f(\tilde t_0)}$, $e^{nf(t_0)}$, $\widetilde Q_j$ and Q_j admit series expansions into the powers of $n^{-1/2}$:

$$e^{n\tilde f(\tilde t_0)} = \exp\left\{-\psi_1^2\left(kx_1 + \frac{2k-1}{2}x_2\right)^2\right\}\left(1 + \sum_{j=1}^{\infty} \frac{1}{n^{j/2}} \widetilde m_j(k, x_1, x_2)\right), \tag{15}$$

$$e^{nf(t_0)} = \exp\left\{-\psi_1^2\left((x_1+x_2)x + kx_1 + \frac{2k-1}{2}x_2\right)^2\right\}\left(1 + \sum_{j=1}^{\infty} \frac{1}{n^{j/2}} m_j(k, x_1, x_2)\right), \tag{16}$$

$$\widetilde{Q}_j = \sum_{j=0}^{\infty} \frac{1}{n^{j/2}} \widetilde{M}_j(k, x_1, x_2), \tag{17}$$

$$Q_j = \sum_{j=0}^{\infty} \frac{1}{n^{j/2}} M_j(k, x_1, x_2). \tag{18}$$

Note that $\widetilde{m}_j(k, x_1, x_2)$, $m_j(k, x_1, x_2)$, $\widetilde{Q}_j$, and Q_j are polynomials of degree at most $3j$ in each variable. Insert the expressions (15)–(18) into (11) and (12), multiply the series, and regroup the terms to combine them like powers of $n^{-1/2}$. As a result, we obtain expansions of the form

$$I_1(n) = e^{-\psi_1^2(kx_1+\frac{2k-1}{2}x_2)^2} \sum_{l=1}^{m} \left\| \sum_{j=0}^{q-1} \frac{1}{n^{(j+1)/2}} \widetilde{L}_j(k, x_1, x_2) + \frac{1}{n^{(q+1)/2}} \Psi_q^{(1)}(k, x_1, x_2) \right\|_{sl}, \tag{19}$$

$$\begin{aligned} I_2(n) = \sum_{l=1}^{m} \Bigg\| \sum_{j=0}^{q-1} \frac{1}{n^{(j+1)/2}} \int_0^1 \widetilde{L}_j(k, x_1, x_2) e^{-\psi_1^2((x_1+x_2)x+kx_1+\frac{2k-1}{2}x_2)^2}\, dx \\ + \frac{1}{n^{(q+1)/2}} \int_0^1 \Psi_q^{(2)}(k, x, x_1, x_2) e^{-\psi_1^2((x_1+x_2)x+kx_1+\frac{2k-1}{2}x_2)^2}\, dx \Bigg\|_{sl}. \end{aligned} \tag{20}$$

It can be shown that $\Psi_q^{(1)}(k, x_1, x_2)$ and the last integral in (20) are bounded. In the case $m = 1$ this was done in [7]. Write

$$\begin{aligned} I_1(n) + (a+b)I_2(n) = I_1(n) + (x_1+x_2)\sqrt{n} I_2(n) \\ \equiv \sum_{l=1}^{m} \left\| \sum_{j=0}^{q-1} \frac{1}{n^{j/2}} U_j(k, x_1, x_2) + \frac{1}{n^{q/2}} \Psi_q(k, x_1, x_2) \right\|_{sl}. \end{aligned} \tag{21}$$

The main term in the expansion (21) is equal to

$$\begin{aligned} \sum_{l=1}^{m} \|U_0(k, x_1, x_2)\|_{sl} = \sum_{l=1}^{m} \left\| \frac{2\psi_1(x_1+x_2)}{\sqrt{\pi}} \int_0^1 e^{-\psi_1^2((x_1+x_2)x+kx_1+\frac{2k-1}{2}x_2)^2} dx I(0) l(0) \right\|_{sl} \\ = 2\left(\Phi_{0,1}\left(\frac{\psi_1}{\sqrt{2}}(2(k+1)(x_1+x_2)-x_2) \right) - \Phi_{0,1}\left(\frac{\psi_1}{\sqrt{2}}(2k(x_1+x_2)-x_2) \right) \right), \end{aligned} \tag{22}$$

where $\Phi_{0,1}(x)$ is the function of the standard normal distribution.

Summarizing the above, we obtain the following

Theorem 5. *Let $k = \text{const}$, $k \geq 1$, $a = x_1\sqrt{n}$, and $b = x_2\sqrt{n}$ and let $n \to \infty$. Then for $q \geq 1$*

$$\mathbb{P}\big(\eta_n^{(1)} = k / \varkappa_0 = s\big) = \sum_{l=1}^{m} \left\| \sum_{j=0}^{q-1} \frac{1}{n^{j/2}} U_j(k, x_1, x_2) + \frac{1}{n^{q/2}} \Psi_q(k, x_1, x_2) \right\|_{sl}, \tag{23}$$

where $U_j(k, x_1, x_2)$ and $\Psi_q(k, x_1, x_2)$ are determined by (15)–(20), and the main term of the expansion (23) is computed in (22).

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