Siberian Mathematical Journal, Vol. 37, No. 4, 1996.
Translated from:
Sibirskiĭ Matematicheskiĭ Zhurnal, Vol. 37, No. 4, pp. 919–931.

ESTIMATES FOR THE ACCURACY OF COUPLING IN THE CENTRAL LIMIT THEOREM

A. I. SAKHANENKO

1. Introduction

Let $\xi_1, \dots, \xi_N$ be independent random variables such that

$$\forall j\ \mathbf{M}\xi_j = 0, \quad 0 < B^2 = \sum \mathbf{D}\xi_j < \infty. \tag{1.1}$$

(Here and in the sequel, the symbol $\sum$ indicates that summation is performed over the variable j varying from 1 to N.) Put

$$S = \sum \xi_j, \quad L_\alpha = B^{-\alpha} \sum \mathbf{M}|\xi_j|^\alpha. \tag{1.2}$$

By the central limit theorem, under certain assumptions the distribution of the random variable S/B is close to the distribution of the random variable Z having the standard normal law

$$\mathbf{P}(Z < x) = \Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-y^2/2)\, dy. \tag{1.3}$$

The problem of estimating closeness between the distributions of S/B and Z was studied by many authors in various directions (see, for instance, [1, 2]). In the present article we obtain estimates for the difference

$$\Delta = |S/B - Z| \tag{1.4}$$

under the assumption that the random variables S and Z are realized in a special way on a common probability space. We prove among other results that, with S and Z suitably constructed on some probability space, the estimate

$$\mathbf{M}\Delta^\alpha \le (C\alpha)^\alpha \sum \mathbf{M} \min\{|\xi_j/B|^{\alpha+2},\ |\xi_j/B|^\alpha\} \tag{1.5}$$

holds for $\alpha \ge 2$, where C is some positive absolute constant. In particular, in this case we have

$$\mathbf{M}\Delta^\alpha \le (C\alpha)^\alpha L_{\alpha+2} \quad \forall\alpha \ge 2. \tag{1.6}$$

In § 2 we also present a series of assertions stronger than (1.5) and (1.6).

To illustrate, we consider an important particular case in which $\xi_j = \zeta_j/n^{1/2}$, where $\zeta_1,\ \zeta_2,\ \dots$ are independent random variables such that

$$\forall j \quad \mathbf{M}\zeta_j = 0, \quad \mathbf{D}\zeta_j = 1, \quad \mathbf{M}|\zeta_j|^\alpha \le \rho_\alpha. \tag{1.7}$$

Date: submitted January 17, 1996.

In this case we introduce the notations

$$S_n = \sum_{j=1}^{n} \zeta_j, \quad \Delta_n = |S_n/n^{1/2} - Z|.$$

Under the above assumptions, (1.6) implies that

$$\mathbf{M}\Delta_n^\alpha \le (C\alpha)^\alpha \rho_{\alpha+2}/n^{\alpha/2} \quad \forall \alpha \ge 2. \tag{1.8}$$

Thus, with (1.7) valid, we have

$$\mathbf{M}\Delta_n^\alpha = O(n^{-\alpha/2}) \quad \text{for } \rho_{\alpha+2} < \infty.$$

We still distinguish a more particular case as compared to (1.7) in which

$$\mathbf{P}(\zeta_j = 1) = \mathbf{P}(\zeta_j = -1) = 1/2. \tag{1.9}$$

Since $n + S_n$ takes only even values, the event $\{n^{1/2}\Delta_n < 1/2\}$ implies one of the events

$$A_{k,n} = \{|(n+Z) - 2k| < 1/2\}, \quad k = 0, 1, \dots, n.$$

By the local limit theorem (see [1, p. 231] and [2, p. 152]),

$$2\sum_{k=0}^{n} \mathbf{P}(A_{k,n}) - 1 = \sum_{k=0}^{n} \big(2\mathbf{P}(A_{k,n}) - \mathbf{P}(n + S_n = 2k)\big) \to 0$$

as $n \to \infty$. Therefore, with (1.9) valid, we have

$$\mathbf{P}(\Delta_n \ge n^{-1/2}/2) \ge 1 - \sum_{k=0}^{n} \mathbf{P}(A_{k,n}) \to 1/2 \quad \text{as } n \to \infty.$$

From the last relation we infer unimprovability of the exponent of n in (1.8).

Estimates for $\mathbf{M}\Delta^\alpha$ have played an important part in studying estimates for the convergence rate in the invariance principle when Ju. V. Prohorov's method is used of constructing a common probability space (see [3, 4]). In particular, in [3] Ju. V. Prohorov established estimate (1.6) (to within the logarithm) for $\alpha = 1$. The inference of the estimate essentially used the equality

$$\mathbf{M}\Delta = \int_{-\infty}^{\infty} |\mathbf{P}(S/B < x) - \mathbf{P}(Z < x)|\, dx.$$

In the article [4], the Ebralidze inequality [5]

$$\mathbf{M}\Delta^\alpha \le 2^{\alpha-1}\alpha \int_{-\infty}^{\infty} |x|^{\alpha-1} |\mathbf{P}(S/B < x) - \mathbf{P}(Z < x)|\, dx \tag{1.10}$$

was used in the derivation of estimates for $\mathbf{M}\Delta^\alpha$ for $\alpha > 1$.

However, the use of (1.10) with $\alpha > 1$ in the problem under study leads to a very rough estimate. Indeed, in the particular case of (1.9) the right-hand side of (1.10) is $O(n^{-1/2})$ instead of the true estimate $O(n^{-\alpha/2})$ which can be extracted from (1.6).

In the present article, the derivation of estimates for $\mathbf{M}\Delta^\alpha$ is based on straightforward estimation of Δ. An essential part is played by the estimates in the central limit theorem which account for large deviations and were obtained by the author in [6].

Let us agree that the symbol □ signifies the end of a proof and that multiplication always precedes division, so that $ab/2cd = (ab)/(2cd)$.

2. The Main Results

Given an arbitrary real h, put

$$L(h) = B^{-3} \sum \mathbf{M}|\xi_j|^3 e^{|h\xi_j|}. \tag{2.1}$$

As shown in [6], $L(h)$ is a natural dominant for the Lyapunov ratio of the third order for the "conjugate distributions"; i.e., the distributions resulting from applying the Cramér transformation. Below in the condition (2.3) of Theorem 1 we assume that this quantity is not too large. Also, we need one more modification of the Lyapunov ratio:

$$L(\alpha, h) = B^{-3} \sum \mathbf{M}|\xi_j|^3 \exp\bigl(\min\bigl\{16\alpha\xi_j^2/B^2, |h\xi_j|\bigr\}\bigr). \tag{2.2}$$

Throughout the article, we assume that the random variables $\{\xi_j\}$ are independent and satisfy (1.1), with S, Z, and Δ defined by (1.2)–(1.4).

Theorem 1. *Let $\lambda > 0$ be such that*

$$60\lambda BL(2\lambda) \le 1. \tag{2.3}$$

Then there is a random variable Z defined on a common probability space with S, having the standard normal law, and such that the following inequality holds for $\alpha \ge 1$:

$$\mathbf{M}\Delta^\alpha e^{\lambda\Delta/8} \le 5(60\alpha)^\alpha L^\alpha(\alpha, 2\lambda) + 8(3\alpha)^\alpha(\lambda B)^{-\alpha} e^{-\lambda^2 B^2/8}. \tag{2.4}$$

We bring one simple corollary in the case when the variables $\{\xi_j\}$ are bounded; i.e.,

$$\forall j \ \mathbf{P}(|\xi_j| \le y) = 1, \quad y < \infty. \tag{2.5}$$

Corollary 1. *Suppose that (2.5) holds. Then (2.3) and hence the claim of Theorem 1 are valid for all $\alpha \ge 1$ and $0 \le \lambda \le \lambda_1$, where*

$$\lambda_1 \equiv (4y)^{-1} \ln(1 + y/15BL_3). \tag{2.6}$$

Since $L_3 \le y/B$ by (2.5) and (1.2); therefore,

$$\lambda_1 \ge (4y)^{-1} \ln(1 + 1/15) \ge (64y)^{-1}. \tag{2.7}$$

Thus, in view of (2.7) estimate (2.4) always takes place with $\lambda = (64y)^{-1}$. However, such choice of λ may fail to be best possible.

We now waive the assumption (2.5) of boundedness of the random variables $\{\xi_j\}$.

Corollary 2. *Let*

$$L(\alpha, \infty) = B^{-3} \sum \mathbf{M}|\xi_j|^3 e^{16\alpha\xi_j^2/B^2} < \infty. \tag{2.8}$$

Then there exists a random variable Z satisfying (1.3) and such that

$$\mathbf{M}\Delta^\alpha e^{\lambda\Delta/8} \le 8(C_0\alpha)^\alpha L^\alpha(\alpha, \infty) \tag{2.9}$$

for all $\alpha \ge 1$ and $0 \le \lambda \le \lambda_2$, where

$$\lambda_2^2 \equiv 4\alpha B^{-2} \ln\bigl(1 + 4C_0^{-2}\alpha^{-1} L^\alpha(\alpha, \infty)\bigr) \tag{2.10}$$

and C_0 is some positive absolute constant.

Now, we apply Corollary 2 to truncated random variables. We put

$$\begin{gathered} L_\alpha(a,b) = B^{-\alpha} \sum \mathbf{M}\{|\xi_j|^\alpha;\ a \le |\xi_j| < b\}, \\ L_*(\alpha,y) = B^{-3} \sum \mathbf{M}\{|\xi_j|^3 e^{25\alpha\xi_j^2/B^2};\ |\xi_j| < y\}, \\ g_{\alpha,\beta,\lambda,K}(x) = \min\{|x|^\alpha e^{\lambda|x|/8};\ K|x|^\beta\}, \\ \lambda_0^2(\alpha,y) = \alpha B^{-2} \ln\bigl(1 + C_0^{-2}\alpha^{-1}L_*^{-2}(\alpha,y)\bigr). \end{gathered} \tag{2.11}$$

Theorem 2. *Suppose that* $0 \le y, K \le \infty$ *and*

$$\alpha \ge 1, \quad \beta \ge 2, \quad 0 \le \lambda \le \lambda_0(\alpha,y). \tag{2.12}$$

Then there exists a random variable Z *satisfying (1.3) and such that*

$$\begin{gathered} \mathbf{M} g_{\alpha,\beta,\lambda,K}(\Delta) \le 8(2C_0\alpha)^\alpha L_*^\alpha(\alpha,y) \\ +4(96\beta)^{\beta/2} K L_2^{\beta/2}(y,\infty) + (4\beta)^\beta K L_\beta(y,\infty), \end{gathered} \tag{2.13}$$

where the absolute constant C_0 *is the same as in Corollary 2.*

Corollary 3. *If* $\alpha \ge \beta \ge 2$ *then there exists a random variable* Z *having the standard normal law and such that*

$$\mathbf{M}\min\{\Delta^\alpha,\, \Delta^\beta\} \le (C\alpha)^\alpha L_3^\alpha(0, B/\alpha^{1/2}) + (C\alpha\beta)^{\beta/2} L_\beta(B/\alpha^{1/2},\infty), \tag{2.14}$$

where $C \in (0,\infty)$ *is some absolute constant.*

Note that we choose the truncation level equal to $B/\alpha^{1/2}$ in Corollary 3 in order to obtain in (2.14) a constant of the form $(C\alpha)^\alpha$ which has the same order of growth in α as the corresponding constant in (2.13). We now choose a more natural truncation level B.

Corollary 4. *Under the conditions of Corollary 3, the random variable* Z *defined therein satisfies the inequality*

$$\mathbf{M}\min\{\Delta^\alpha, \Delta^\beta\} \le (C\alpha)^\alpha L_{\alpha+2}(0,B) + (C\alpha\beta)^{\beta/2} L_\beta(B,\infty). \tag{2.15}$$

Estimate (1.5) mentioned in the Introduction ensues from (2.15) with $\alpha = \beta$.

3. Proof of Theorem 1

The proof of Theorem 1 splits into a succession of lemmas, of which the central ones are Lemmas 1, 2, and 3. The proof is based on using quantile transformations (see Lemma 1) and estimates for them obtained in [6] (see Lemma 2).

Put

$$F(x) = \mathbf{P}(S < x), \quad F^{-1}(t) = \sup\{x : F(x) \le t\}.$$

Lemma 1. *There exits a random variable* Z *having the standard normal law and such that*

$$F(S) \le \Phi(Z) \le F(S+0). \tag{3.1}$$

Proof. From the familiar properties of quantile transformations we infer that, given a random variable Z^* with the standard normal law, the random variable $S^* = F^{-1}(\Phi(Z^*))$ has $F(x)$ as its distribution function. Conversely, as shown by A. V. Skorohod in [7], given a random variable S, it is possible to enlarge the probability space and find a random variable Z so that the pair (Z,S) is distributed identically with the pair (Z^*,S^*). In particular, in this case we have $S = F^{-1}(\Phi(Z))$; i.e., (3.1) holds. □

Unless the contrary is indicated, we henceforth assume the following conditions to be satisfied:

$$B = 1, \quad 60\lambda L(2\lambda) \leq 1, \quad \alpha \geq 1. \tag{3.2}$$

Lemma 2. *If*

$$\lambda \geq 1 \quad \textit{and} \quad |S| < \lambda \tag{3.3}$$

then

$$|S - Z| \leq (S^2 + 60)L(2S) \tag{3.4}$$

and in particular

$$\lambda|S - Z| \leq 1 + S^2/60. \tag{3.5}$$

Proof. Since $\Phi(x)$ is monotone, to each x there is a unique solution $\Psi(x)$ of the equation

$$F(x) = \Phi(\Psi(x)). \tag{3.6}$$

As proven in Corollary 6 of the author's article [6], under the conditions

$$\lambda \geq 1 \quad \text{and} \quad |x| \leq \lambda \tag{3.7}$$

the following inequality holds:

$$|\Psi(x) - \alpha(x)| \leq 60L(2x). \tag{3.8}$$

Here $\alpha(x)$ is some function for which Theorem 4 of [6] gives the estimate

$$|\alpha(x) - x| \leq x^2 K(2x)/4; \tag{3.9}$$

furthermore, it is proven in Corollary 2 of [6] that

$$K(h) \leq 4L(h) \quad \forall h \geq 0. \tag{3.10}$$

Inserting (3.9) and (3.10) in (3.8), we see that, with (3.7) valid, the following inequality holds:

$$|\Psi(x) - x| \leq (x^2 + 60)L(2x). \tag{3.11}$$

Comparing (3.1) and (3.6), we find that

$$\Psi(S) \leq Z \leq \Psi(S + 0). \tag{3.12}$$

To achieve the claim (3.4) of Lemma 2, we must put $x = S$ in (3.7) and (3.11) and make use of (3.12).

Inequality (3.5) follows from (3.4) and (3.3). □

Put

$$\begin{gathered} g(x) = |x|^\alpha e^{\lambda|x|/8} \quad \text{for } \alpha \geq 1 \text{ and } \lambda > 0, \\ E_0(u) = \mathbf{M}\{g(S - Z);\ |S| < u\}, \\ E_\pm(u) = \mathbf{M}\{g(S - Z);\ \pm S \geq u\}. \end{gathered} \tag{3.13}$$

We further proceed with estimating the quantity

$$\mathbf{M}g(\Delta) = \mathbf{M}g(S - Z) = E_0(u) + E_+(u) + E_-(u) \tag{3.14}$$

in several steps.

Lemma 3. *If a function $k(y) > 0$ satisfies the condition*

$$E(y) \equiv \mathbf{M}\{\exp(2\alpha|S||y| + S^2/50); |S| < \lambda\} \leq k^\alpha(y), \tag{3.15}$$

then

$$E_0(\lambda) \leq e^{1/8}(60\alpha K_0)^\alpha \quad \textit{for } K_0 \equiv \sum \mathbf{M}|\xi_j|^3 k(\xi_j). \tag{3.16}$$

Proof. Introduce a probability measure G on the real axis by setting

$$G(A) = K_0^{-1} \sum \mathbf{M}\{|\xi_j|^3 k(\xi_j);\ \xi_j \in A\}. \tag{3.17}$$

In view of (2.1) and (3.17), we then obtain

$$K_0^{-1} L(h) = \int_{-\infty}^{\infty} e^{h|y|} k^{-1}(y) G(dy).$$

By the Hölder inequality, we in particular have

$$K_0^{-\alpha} L^\alpha(h) \le \int_{-\infty}^{\infty} e^{\alpha h|y|} k^{-\alpha}(y) G(dy) \equiv K_*(h). \tag{3.18}$$

From Lemma 2 and (3.13) we deduce that, under the condition (3.3),

$$\begin{aligned} g(S-Z) &\le (60+S^2)^\alpha L^\alpha(2S) \exp(1/8 + S^2/8\cdot 60) \\ &\le e^{1/8}(60\alpha)^\alpha L^\alpha(2S) \exp(S^2/60 + S^2/8\cdot 60). \end{aligned} \tag{3.19}$$

Inserting (3.18) and (3.19) in (3.13), we find that

$$\begin{aligned} E_0(\lambda) &\le e^{1/8}(60\alpha)^\alpha K_0^\alpha \mathbf{M}\{\exp(S^2/50)\cdot K_*(2S);\ |S| < \lambda\} \\ &= e^{1/8}(60\alpha K_0)^\alpha \int_{-\infty}^{\infty} E(y) k^{-\alpha}(y) G(dy). \end{aligned}$$

Now, (3.16) is immediate from the preceding relation and (3.15). □

Lemma 4. *If* $|h| \le 2\lambda$ *then*

$$\mathbf{M}e^{hS} \le \exp(h^2/2 + |h|^3 L(2\lambda)/6). \tag{3.20}$$

Proof. For $\xi = \xi_j$ and $|h| \le 2\lambda$, we have

$$\begin{aligned} \mathbf{M}e^{h\xi} &\le \mathbf{M}(1 + h\xi + h^2\xi^2/2 + |h\xi|^3 e^{|h\xi|}/6) \le 1 + h^2\mathbf{D}\xi/2 + |h|^3\mathbf{M}|\xi|^3 e^{2\lambda|\xi|}/6 \\ &\le \exp(h^2\mathbf{D}\xi/2 + |h|^3\mathbf{M}|\xi|^3 e^{2\lambda|\xi|}/6). \end{aligned}$$

Next,

$$\mathbf{M}e^{hS} = \prod_{j=1}^{n} \mathbf{M}e^{h\xi_j} \le \exp\Big(h^2 \sum \mathbf{D}\xi_j/2 + |h|^3 \sum \mathbf{M}|\xi_j|^3 e^{2\lambda|\xi_j|}/6\Big).$$

It is easy to verify that the right-hand side of this relation coincides with (3.20). □

From (3.2) and (3.20) we deduce that

$$\mathbf{M}e^{hS} \le e^{(1+1/90)h^2/2} \quad \text{for} \quad |h| \le 2\lambda. \tag{3.21}$$

Lemma 5. *If* $0 \le x \le 3\lambda$ *then*

$$\mathbf{P}(|S| \ge x) \le 2e^{-x^2/3}. \tag{3.22}$$

Proof. From (3.21) with $x > 0$ and $0 \le h \le 2\lambda$ we have

$$\mathbf{P}(\pm S \ge x) \le e^{-hx}\mathbf{M}e^{\pm hS} \le \exp(3h^2/4 - hx). \tag{3.23}$$

Inserting $h = 2x/3 \le 2\lambda$ in (3.23), we easily obtain the required assertion. □

Lemma 6. *Condition (3.15) holds for*

$$k(y) = 4^{1/\alpha} \exp(\min\{16\alpha y^2,\, 2\lambda|y|\}). \tag{3.24}$$

Proof. Consider the monotone function

$$f(x) = \exp(2\alpha|y||x| + x^2/50) \tag{3.25}$$

and successively use (3.15) and (3.22) to obtain

$$\begin{aligned} E(y) &\le \mathbf{M} f(\min\{|S|,\, \lambda\}) = f(0) + \int_0^\lambda \mathbf{P}(|S| > x)\, df(x) \\ &\le f(0) + \int_0^\lambda 2e^{-x^2/3} df(x) \le \int_0^\infty f(\min\{x, \lambda\})\, d(-2e^{-x^2/3}). \end{aligned} \tag{3.26}$$

From (3.25) and (3.26) we find that

$$E(y) \le e^{2\alpha\lambda|y|} \int_0^\infty e^{x^2/50}\, d(-2e^{-x^2/3}) \le e^{2\alpha\lambda|y|} \int_0^\infty (4x/3) e^{x^2/6}\, dx = 4e^{2\alpha\lambda|y|}. \tag{3.27}$$

Furthermore, since $2\alpha|y|x \le 16(\alpha|y|)^2 + x^2/16$, it follows that

$$\begin{aligned} E(y) &\le \int_0^\infty f(x) d(-2e^{-x^2/3}) \le \int_0^\infty e^{16(\alpha|y|)^2 + x^2/16 + x^2/50 - x^2/3} (4x/3)\, dx \\ &\le e^{16(\alpha|y|)^2} \int_0^\infty e^{-x^2/6} (4x/3)\, dx = 4e^{16\alpha^2 y^2}. \end{aligned} \tag{3.28}$$

Now, (3.27) and (3.28) yield (3.15) with $k(y)$ defined by (3.24). □

Lemma 7. *If* $u \ge \max\{\lambda, 1\}$ *then*

$$E_\pm(u) \le 2(3\alpha)^\alpha \lambda^{-\alpha} e^{-\lambda u/8}. \tag{3.29}$$

Proof. First of all, observe that

$$\mathbf{P}(\pm S \ge 1) \le 1/2. \tag{3.30}$$

This fact follows from the Cantelli inequality which gives

$$\mathbf{P}(\pm S \ge x) \le \mathbf{D}S/(\mathbf{D}S + x^2).$$

Thus, (3.1), (3.30), and the symmetry of the random variable Z imply that

$$\operatorname{sign} Z = \operatorname{sign} S \quad \text{for } |S| > 1. \tag{3.31}$$

Consequently, for $u \ge 1$

$$E_+(u) \le \mathbf{M}\{g(S);\, S \ge u\} + \mathbf{M}\{g(Z);\, Z > S \ge u\}. \tag{3.32}$$

We now use the relation

$$g(x) = |x|^\alpha e^{\lambda|x|/8} \le (3\alpha)^\alpha \lambda^{-\alpha} e^{\lambda|x|/4} \tag{3.33}$$

which ensues from the chain of the inequalities $y = 8(y/8) \le 8e^{y/8-1} \le 3e^{y/8}$ for $y = \lambda|x|/\alpha$. By (3.33)

$$g(S) \le (3\alpha)^\alpha \lambda^{-\alpha} e^{-\lambda u/2} e^{(3/4)\lambda S} \quad \text{for} \quad S \ge u. \tag{3.34}$$

From (3.21) and (3.34) for $\lambda \le u$ we obtain

$$\begin{aligned} &\mathbf{M}\{g(S);\, S \ge u\} \le (3\alpha)^\alpha \lambda^{-\alpha} e^{-\lambda u/2} \mathbf{M} e^{(3/4)\lambda S} \\ &\le (3\alpha)^\alpha \lambda^{-\alpha} e^{-\lambda u/2} e^{(4/3)(3/4)^2\lambda^2/2} \le (3\alpha)^\alpha \lambda^{-\alpha} e^{-\lambda u/8}. \end{aligned} \tag{3.35}$$

Replacing S in (3.35) with the normally distributed variable Z, we can easily find that

$$\mathbf{M}\{g(Z);\, Z \ge u\} \le (3\alpha)^\alpha \lambda^{-\alpha} e^{-\lambda u/8}. \tag{3.36}$$

Deriving (3.36), we must use the identity

$$\mathbf{M} e^{hZ} = e^{h^2/2} \quad \forall h \tag{3.37}$$

rather than (3.21).

The required inequality (3.29) for $E_+(u)$ follows from (3.32), (3.35), and (3.36). Replacing S and Z in (3.31)–(3.37) with $-S$ and $-Z$ respectively, we easily obtain an analogous estimate for $E_-(u)$. □

Lemma 8. *If* $\lambda \ge 1$ *then*

$$\begin{aligned} &\mathbf{M} g(S - Z) \le 4(3\alpha)^\alpha \lambda^{-\alpha} e^{-\lambda^2/8} \\ &+5(60\alpha)^\alpha \Big(\sum \mathbf{M}|\xi_j|^3 \exp\big(\min\{16\alpha\xi_j^2;\ 2\lambda|\xi_j|\}\big)\Big)^\alpha. \end{aligned} \tag{3.38}$$

To prove this assertion, we have to use (3.14) with $u = \lambda$ while inserting in it the estimates for $E_\pm(\lambda)$ given by Lemma 7 and the estimate for $E_0(\lambda)$ obtained in Lemmas 3 and 6.

Lemma 9. *If* $\lambda \le 1$ *then*

$$\mathbf{M} g(S - Z) \le 8(3\alpha)^\alpha \lambda^{-\alpha} e^{-\lambda^2/8}. \tag{3.39}$$

Proof. First, observe that

$$|S - Z| \le 1 + |Z| \quad \text{for} \quad |S| \le 1.$$

From this relation and (3.33) we have

$$\begin{aligned} &E_0(1) \le \mathbf{M} g(1 + |Z|) \le (3\alpha)^\alpha \lambda^{-\alpha} e^{\lambda/4} \mathbf{M} e^{\lambda|Z|/4} \\ &\le (3\alpha)^\alpha \lambda^{-\alpha} e^{\lambda/4} \cdot 2e^{(\lambda/4)^2/2} < 4(3\alpha)^\alpha \lambda^{-\alpha} e^{-\lambda^2/8}. \end{aligned} \tag{3.40}$$

Deriving the last inequality, we have used (3.37) together with the condition $\lambda \le 1$.

Inserting (3.40) and (3.29) with $u = 1$ in (3.14), we come to the required assertion (3.39). □

Lemmas 8 and 9 readily yield the validity of all assertions of Theorem 1 for $B = 1$, because in this case the conditions (3.2) coincide with those in the hypothesis of Theorem 1 and the right-hand sides in (3.38) and (3.39) do not exceed the right-hand side of (2.4).

To eliminate the last constraint, we should replace λ, S, and $\{\xi_j\}$ in these lemmas with λB, S/B, and $\{\xi_j/B\}$ respectively.

4. Proof of Corollaries and Theorem 2

Lemma 10. *Under the condition (2.5), we have (2.3) for $\lambda = \lambda_1$, where λ_1 is defined by (2.6).*

Proof. For $\lambda > 0$ and $\xi = \xi_j$, we have

$$4\lambda y \mathbf{M}|\xi|^3 e^{2\lambda|\xi|} \le 4\lambda y e^{2\lambda y}\mathbf{M}|\xi|^3 \le (e^{4\lambda y} - 1)\mathbf{M}|\xi|^3.$$

Consequently, definition (2.1) of $L(2\lambda)$ implies that

$$60\lambda B L(2\lambda) \le (15B/y)(e^{4\lambda y} - 1)L_3. \tag{4.1}$$

Inserting the value $\lambda = \lambda_1$ of (2.6) in (4.1), we arrive at the required inequality (2.3). □

Corollary 1 is immediate from Lemma 10.

Lemma 11. *Under the condition (2.8), we have (2.3) for $\lambda = \lambda_2$, where λ_2 is defined by (2.10).*

Proof. We use the inequalities

$$|\xi|^3 e^{2\lambda|\xi|} \le |\xi|^3 e^{16\alpha\xi^2/B^2 + \lambda^2 B^2/16\alpha} \quad \text{for } \xi = \xi_j, \tag{4.2}$$

$$2xe^x \le e^{2x} - 1 \quad \text{for } x = (\lambda^2 B)^2/8\alpha. \tag{4.3}$$

From (4.2), (2.1) and (2.2) we obtain

$$L(2\lambda) \le e^{\lambda^2 B^2/16\alpha} L(\alpha, \infty). \tag{4.4}$$

It follows from (4.4) and (4.3) that

$$\begin{aligned}(60\lambda B L(2\lambda))^2 &\le 60^2 L^2(\alpha,\infty)(\lambda B)^2 e^{\lambda^2 B^2/8\alpha} \\ &\le 120^2 \alpha L^2(\alpha,\infty)(e^{\lambda^2 B^2/4\alpha} - 1).\end{aligned} \tag{4.5}$$

The validity of (2.3) for $\lambda = \lambda_2$ ensues from (2.10) and (4.5) with $C_0 = 240$. □

Proof of Corollary 2. We simplify notation by setting $\lambda = \lambda_2$, $C_0 = 240$, and $x_* = 2^{-2}C_0^2\alpha L^2(\alpha,\infty)$. It is easy to verify that in this case $\lambda^2 B^2 = 4\alpha \ln\bigl(1+x_*^{-1}\bigr) \ge 4\alpha(1+x_*)^{-1}$, and hence

$$\begin{aligned}(\lambda B)^{-\alpha} e^{-\lambda^2 B^2/8} \le (4\alpha)^{-\alpha/2}(1+x_*)^{\alpha/2}\exp\bigl(-(\alpha/2)\ln\bigl(1+x_*^{-1}\bigr)\bigr) = \\ (4\alpha)^{-\alpha/2} x_*^{\alpha/2} = (C_0/4)^\alpha L^\alpha(\alpha,\infty).\end{aligned} \tag{4.6}$$

Inserting (4.6) in (2.4), we immediately obtain (2.9) on observing only that $L(\alpha, 2\lambda) \le L(\alpha,\infty)$ and $5 \cdot 60^\alpha \le 2 \cdot C_0^\alpha$, $8 \cdot 3^\alpha \cdot (C_0/4)^\alpha \le 6C_0^\alpha$. □

We turn to proving Theorem 2. We fix a number $0 \le y < \infty$ and proceed with a series of lemmas.

Lemma 12. *There is sequence of independent pairs $\bigl\{\xi_j',\ \xi_j''\bigr\}$ of random variables such that for all j and all $\beta \ge 1$*

$$\xi_j = \xi_j' + \xi_j'', \quad \mathbf{M}\xi_j' = \mathbf{M}\xi_j'' = 0, \quad \mathbf{P}\bigl(\bigl|\xi_j'\bigr| < y\bigr) = 1, \quad \xi_j' \cdot \xi_j'' = 0, \tag{4.7}$$

$$\mathbf{M}\bigl|\xi_j''\bigr|^\beta \le 2\mathbf{M}\bigl\{\bigl|\xi_j\bigr|^\beta;\ |\xi_j| \ge y\bigr\}. \tag{4.8}$$

If ξ_j is symmetrically distributed then we should take as ξ_j' the two-sided truncation of ξ_j at levels $\pm y$. If ξ_j has a continuous distribution function then we should take ξ_j' to be an asymmetric truncation as described by A. V. Skorohod in [8]. In the general case we additionally have to use randomization (see [9]).

Put

$$S' = \sum \xi_j', \quad S'' = \sum \xi_j'',$$
$$B' = (\mathbf{D}S')^{1/2}, \quad B'' = (\mathbf{D}S'')^{1/2}, \tag{4.9}$$
$$\Delta' = |S'/B' - Z|, \quad \Delta'' = |(B'/B)Z - Z|.$$

Observe that, by virtue of (4.7)–(4.9),

$$B^2 - (B')^2 = (B'')^2 \le 2B^2 L_2(y,\infty). \tag{4.10}$$

Lemma 13. *Suppose that*

$$B' \ge (4/5)B. \tag{4.11}$$

Then there is a random variable Z satisfying (1.3) and such that

$$\mathbf{M}g_{\alpha,\lambda}(\Delta') \le 8(2C_0\alpha L_*(\alpha,y))^\alpha \equiv \delta', \tag{4.12}$$

where $\alpha \ge 1$ and

$$g_{\alpha,\lambda}(x) = |x|^\alpha e^{\lambda|x|/8}, \quad 0 \le \lambda \le 2\lambda_0(\alpha,y). \tag{4.13}$$

Proof. We make use of the assertion of Corollary 2, replacing $\{\xi_j\}$ with $\{\xi_j'\}$ in it. Having this replacement performed in (2.8)–(2.10), we obtain

$$L(\alpha,\infty) \le 2L_*(\alpha,y) \quad \text{and} \quad \lambda_2 \ge 2\lambda_0(\alpha,y), \tag{4.14}$$

if we first observe that the validity of (4.11) implies that

$$16\big(\xi_j'/B'\big)^2 \le 25\big(\xi_j'/B\big)^2,$$
$$\big|\xi_j'\big|^3/(B')^3 \le (5/4)^3\big|\xi_j'\big|^3/B^3 \le 2\big|\xi_j'\big|^3/B^3.$$

Thus, Corollary 2 and (4.14) yield the claim of Lemma 13. □

From now on, we assume that B' satisfies the condition (4.11) of Lemma 13. To estimate Δ, we will use the inequality

$$\Delta \le (B'/B)\Delta' + \Delta'' + |S''/B| \tag{4.15}$$

which is immediate from (4.9) and the equality $S = S' + S''$.

Lemma 14. *The following inequality holds for all $y \ge 0$ and $\beta \ge 2$:*

$$\mathbf{M}(\Delta'' + |S''/B|)^\beta \le \delta'', \tag{4.16}$$

where

$$\delta'' = (2\beta)^\beta L_\beta(y,\infty) + 4(24\beta L_2(y,\infty))^{\beta/2}. \tag{4.17}$$

Proof. We use the inequality

$$\mathbf{M}|S''|^\beta \le c^\beta \sum \mathbf{M}|\xi_j''|^\beta + 2e^c c^{\beta/2}(B'')^\beta \quad \text{for } c = 1 + \beta/2 \le \beta \tag{4.18}$$

which was proven in [10]. By (4.9), (1.3), and (4.18), we have

$$\mathbf{M}(\Delta'')^\beta = (1 - B'/B)^\beta \mathbf{M}|Z|^\beta \le 2e^{1+\beta/2}\beta^{\beta/2}(1 - B'/B)^\beta. \tag{4.19}$$

Observe that

$$1 - B'/B = (B^2 - (B')^2)/(B^2 + B'B) \le (5/9)(B'')^2/B^2 \le 5B''/9B. \tag{4.20}$$

Deriving (4.20), we have used (4.10).

Now, we apply the inequality

$$|x+z|^\beta \le 2^{\beta-1}(|x|^\beta + |z|^\beta), \quad \beta \ge 2, \tag{4.21}$$

with $x = \Delta''$ and $z = |S''/B|$. After coarsening constants, (4.18)–(4.21) imply

$$\mathbf{M}(\Delta'' + |S''/B|)^\beta \le (1/2)(2\beta)^\beta B^{-\beta} \sum \mathbf{M}|\xi_j''|^\beta + 4(4e\beta)^{\beta/2}(B''/B^\beta)^\beta. \quad (4.22)$$

Estimating the right-hand side of (4.22) by means of (4.8) and (4.10), we easily obtain (4.16) for δ'' defined by (4.17). □

Lemma 15. *The claim of Theorem 2 is valid under the conditions (2.12) and (4.11).*

Proof. Using the notations of (2.11) and (4.13), we have

$$g_{\alpha,\beta,\lambda,K}(x+z) \le g_{\alpha,\beta,\lambda,K}(2|x|) + g_{\alpha,\beta,\lambda,K}(2|z|),$$

$$g_{\alpha,\beta,\lambda,K}(x) = \min\{g_{\alpha,\lambda}(x),\, K|x|^\beta\}.$$

Therefore,

$$g_{\alpha,\beta,\lambda,K}(x+z) \le 2^\alpha g_{\alpha,2\lambda}(x) + K2^\beta |z|^\beta. \quad (4.23)$$

In (4.23), put

$$x = (B'/B)\Delta' \le \Delta', \quad z = \Delta'' + |S''/B|. \quad (4.24)$$

Since $\Delta \le x + z$ by (4.15), we infer from (4.23), (4.24), and Lemmas 13 and 14 that

$$\mathbf{M}g_{\alpha,\beta,\lambda,K}(\Delta) \le 2^\alpha \delta' + 2^\beta K \delta''. \quad (4.25)$$

To obtain (2.13), we have to insert in (4.25) the expressions for δ' and δ'' given by (4.12) and (4.17). □

Lemma 16. *For all $\beta \ge 2$, the following inequality holds:*

$$\mathbf{M}g_{\alpha,\beta,\lambda,K}(\Delta) \le (1/2)(2\beta)^\beta K L_\beta(0,\infty) + 4(12\beta)^{\beta/2} K. \quad (4.26)$$

Proof. Observe that we always have

$$\mathbf{M}g_{\alpha,\beta,\lambda,K}(\Delta) \le K\mathbf{M}|\Delta|^\beta. \quad (4.27)$$

Moreover,

$$\Delta \le \Delta'' + |S''/B| \quad \text{for } y = 0. \quad (4.28)$$

Successively applying (4.27), (4.28), and (4.22), we obtain (4.26) on observing that $B'' = B$ and $\xi_j'' = \xi_j$ for $y = 0$. □

Proof of Theorem 2. If (4.11) holds then Theorem 2 ensues from Lemma 15.

Now, assume that (4.11) fails. By (4.10), we then have

$$1/3 \le 1 - (4/5)^2 \le 1 - (B'/B)^2 = (B''/B)^2 \le 2L_2(y,\infty).$$

Consequently, in this case

$$1 \le 6L_2(y,\infty) \le (6L_2(y,\infty))^{\beta/2} \quad \text{for } \beta \ge 2. \quad (4.29)$$

Furthermore, by Chebyshev's inequality

$$1 \le 6L_2(y,\infty) \le 6B^{\beta-2}L_\beta(y,\infty)/y^{\beta-2};$$

i.e., in the case under study

$$y^{\beta-2} \le 6B^{\beta-2}L_\beta(y,\infty).$$

Therefore, in view of (2.11)

$$L_\beta(0,\infty) = L_\beta(0,y) + L_\beta(y,\infty) \le y^{\beta-2}B^2/B^\beta + L_\beta(y,\infty) \le 7L_\beta(y,\infty). \quad (4.30)$$

Inserting (4.29) and (4.30) in (4.26), we obtain

$$\mathbf{M}g_{\alpha,\beta,\lambda,K}(\Delta) \le 4(2\beta)^\beta K L_\beta(y,\infty) + 4K(72\beta L_2(y,\infty))^{\beta/2}.$$

Thus, Theorem 2 is also true in the trivial case in which (4.11) fails. □

To derive Corollaries 3 and 4, we need

Lemma 17. *If* $\gamma \geq 0$ *and* $r \geq 1$ *then*

$$L^r_{2+\gamma}(a,b) \leq L_{2+r\gamma}(a,b). \tag{4.31}$$

Proof. By analogy to the proof of Lemma 2 in [1, p. 173], we introduce the probability measure

$$\mu(A) = B^{-2} \sum \mathbf{M}\{\xi_j^2; , \xi_i \in A\}.$$

In this case

$$L_{2+\gamma}(a,b) = B^{-\gamma} \int\limits_{[a,b)} |x|^\gamma \mu(dx). \tag{4.32}$$

Consequently, by Hölder's inequality

$$L^r_{2+\gamma}(a,b) \leq B^{-r\gamma} \int\limits_{[a,b)} |x|^{r\gamma} \mu(dx). \tag{4.33}$$

From (4.32) and (4.33) we deduce (4.31). □

Lemma 18. *If the conditions*

$$y = B/\alpha^{1/2}, \quad \lambda = 0, \quad K = 1, \quad \alpha \geq \beta \geq 2 \tag{4.34}$$

are satisfied then the right-hand side of (2.13) does not exceed

$$2^{-1}(C\alpha L_3(0,y))^\alpha + 2^{-1}\alpha^{-1}(C\alpha\beta)^{\beta/2} L_\beta(y,\infty) \tag{4.35}$$

with $C \leq \max\{16 \cdot 2e^{25} C_0,\ 10 \cdot 96\}$.

Proof. From (2.11) and (4.34) we obtain

$$L_*(\alpha, y) \leq e^{25} L_3(0,y), \tag{4.36}$$

$$L_2(y,\infty) \leq \alpha^{\gamma/2} L_{2+\gamma}(y,\infty) \tag{4.37}$$

for $\gamma \geq 0$. By setting $\gamma = 2(\beta-2)/\beta \geq 0$ in (4.31) and (4.37), we infer that

$$L_2^{\beta/2}(y,\infty) \leq \alpha^{\beta/2-1} L_\beta(y,\infty). \tag{4.38}$$

To reach the required assertion of Lemma 18, we must insert (4.36) and (4.38) in (2.13) and use the inequality $4^\beta \leq \beta^{-1}(96)^{\beta/2}$ while estimating the constants. □

Corollary 3 is immediate from Lemma 18.

Proof of Corollary 4. Assume that (4.34) holds. In this case Lemma 17 yields

$$L_3^\alpha(0,y) \leq L_{2+\alpha}(0,y). \tag{4.39}$$

Furthermore, from definitions (2.11) and Chebyshev's inequality we obtain

$$L_\beta(y,\infty) = L_\beta(y,B) + L_\beta(B,\infty), \quad L_\beta(y,B) \leq (\alpha^{1/2})^{\alpha+2-\beta} L_{\alpha+2}(y,B). \tag{4.40}$$

Inserting (4.39) and (4.40) in (4.35), we come to the desired assertion (2.15). □

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