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Approximations of open queueing networks by reflection mappings

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Abstract

This paper presents a three-step procedure which allows to approximate the queue-length and the busy-time processes associated with open queueing networks. These three approximations are based on reflection mappings and are introduced with explicit estimates of their accuracy. The third one may be treated as approximation by accompanying reflection Brownian motions with rates of convergence.

Keywords: flow networks, diffusion approximations, sample path analysis, open queueing networks, rate of convergence, reflection mappings, reflected Brownian motion.

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1. Introduction

In this paper we consider open queueing networks with general arrival, service and routing processes. The main object of our interest is the dynamics of the queue-length process $\mathbf{Q}(\cdot)$ and busy-time process $\mathbf{B}(\cdot)$. Our study is motivated by the works of M. I. Reiman [12], H. Chen and A. Mandelbaum [3, 4], where diffusion limit theorems for the distributions of these processes were established under the heavy traffic assumptions, and by the papers of L. Horvath [11], H. Chen and A. Mandelbaum [5], where accuracy of approximations for the mentioned processes were investigated using the method of J. Komlos, P. Major and G. Tusnady. Our aim is to obtain new pathwise approximations for the processes $\mathbf{Q}(\cdot)$ and $\mathbf{B}(\cdot)$ which would have analogous accuracy under more general assumptions.

The starting point of our study is the same as in [12, 3, 4, 5]. We use the representation which is due to J. M. Harrison and write it in the form

$$\mathbf{B}(t) = \overline{\mathbf{B}}(t; \mathbf{X}(\cdot)), \quad \mathbf{Q}(t) = \overline{\mathbf{Q}}(t; \mathbf{X}(\cdot)), \tag{1.1}$$

where $\overline{\mathbf{B}}(\cdot\,; \mathbf{X})$ and $\overline{\mathbf{Q}}(\cdot\,; \mathbf{X})$ are some reflection mappings and the process $\mathbf{X}(t) = \overline{\mathbf{X}}(t; \mathbf{B}(\cdot))$ is a linear combination of arrival, service and routing processes, taken at points dependent on $\mathbf{B}(\cdot)$.

The first step in our approach is an analogue of the fluid approximation in [3, 4, 5]. We take a function $\hat{X}_0(\cdot)$ close to the function $\overline{\mathbf{X}}(\cdot\,; 0)$ and put $\mathbf{b}(\cdot) = \overline{\mathbf{B}}(\cdot\,; \hat{X}_0(\cdot))$ as the first approximation for the function $\mathbf{B}(\cdot)$. In the simplest case the components of the vector-function $\mathbf{b}(\cdot)$ equal $\rho_j t$, where ρ_j is the traffic intensity at station j.

For the second step we choose a function $\hat{\mathbf{X}}(\cdot)$ close to the function $\overline{\mathbf{X}}(\cdot\,; \mathbf{b}(\cdot))$ and introduce

$$\hat{\mathbf{B}}(\cdot) = \overline{\mathbf{B}}(\cdot\,; \hat{\mathbf{X}}(\cdot)) \quad \text{and} \quad \hat{\mathbf{Q}}(\cdot) = \overline{\mathbf{Q}}(\cdot\,; \hat{\mathbf{X}}(\cdot)), \tag{1.2}$$

as desirable pathwise approximations for the processes $\mathbf{B}(\cdot)$ and $\mathbf{Q}(\cdot)$.

In the simplest case the process $\hat{\mathbf{X}}(\cdot)$ is a linear combination of sums of random vectors and renewal processes taken at the points t or $\rho_j t$. For this reason we may use the invariance principles [6, 10, 13] for these processes and change the distribution of the process $\hat{\mathbf{X}}(\cdot)$ to the distribution of the accompanying Brownian motion $\hat{\boldsymbol{\xi}}(\cdot)$, with the drift and the covariance matrix determined by the first and the second moments of interarrival and service times and by the routing matrix. At this third step, using the ideas of A. V. Skorohod [14], we may enlarge our probability space and approximate the busy-time process $\mathbf{B}(\cdot)$ and the queue-length process $\mathbf{Q}(\cdot)$ with the diffusion processes $\overline{\mathbf{B}}(\cdot\,; \hat{\boldsymbol{\xi}})$ and $\overline{\mathbf{Q}}(\cdot\,; \hat{\boldsymbol{\xi}})$ respectively.

Earlier the analogous ideas were used in the works of L. Horvath [11], H. Chen and A. Mandelbaum [5]. The present modification of these ideas allow us to obtain explicit nonasymptotic estimates of the remainder terms for corresponding approximations under more general assumptions.

The paper is organized as follows. In Section 2 we present definitions of reflection mappings and their properties, which play a key role in our study. Section 3 contains a standard description of the queueing network that we consider and some useful representations for the processes $\mathbf{B}(\cdot)$ and $\mathbf{Q}(\cdot)$ through the reflection mappings.

In Section 4 we describe rigorously the above-mentioned three steps of our approach. The main purpose of this section is to obtain general estimates for the accuracy of these approximations. This goal is achieved in Theorems 4.1, 4.2 and 4.3.

In Section 5 we illustrate the possibilities of Theorems 4.1, 4.2 and 4.3 by studying a standard queueing network. We have obtained here estimates which, in the corresponding particular cases, have the same unimprovable accuracy as those of [11, 5] but under weaker conditions. For example, the distributions of the arrival and service processes may essentially depend on the traffic. Appendix A contains the proofs of the auxiliary results used in Section 5.

Now we briefly introduce some notations and conventions that we use throughout the paper. The positive part of a number x is denoted by $x^+ = \max\{0, x\}$ and the integer part by $[x]$. For a given function $x(\cdot)$, let

$$\|x(\cdot)\|_t = \sup_{0\le s\le t} |x(s)| \quad \text{and} \quad \beta(\alpha; t, x(\cdot)) = \sup_{0\le s\le t,\, |\delta|\le\alpha} |x(s+\delta) - x(s)| \tag{1.3}$$

be its uniform norm and the maximal oscillation on the interval $[0, t]$, respectively. Denote by $\mathcal{D}[0, T]$ the space of the functions on $[0, T]$ that are right-continuous with finite left limits. As in [4], we consider this space with the uniform norm $\|\cdot\|_T$.

Throughout the paper J denotes either a finite set $J = \{1, 2, ...|J|\}$ of the first positive integers, or a set of all positive integers with $|J| = \infty$. Indices usually run through this set J. If some index can have value 0, then it is specially mentioned. The summation sign with an index means that the sum is taken over J, so

$$\sum_i x_i = \sum_{i\in J} x_i.$$

The bold letters $\mathbf{P}$ and $\mathbf{E}$ stand for the probability and expectation. The other bold letters are used to denote vector-functions from $\mathcal{D}^J[0, T]$. Thus, $\mathbf{x}$ is a vector-function with components $\{x_j(\cdot), j \in J\}$. Vector-functions with two arguments and semicolon between them , as in (1.1), denote non-anticipating mappings with respect to their second argument. This means that for $\mathbf{Z}(\cdot\,; \mathbf{x})$ its components $Z_j(t; \mathbf{x})$ at t are defined uniquely by the values of the functions $\{x_j(\cdot), j \in J\}$ on the interval $[0, t]$.

2. Reflection mappings and their properties

Reflection mappings is the main tool of our study. Necessary facts about them are formulated in this section. Let P be a matrix with nonnegative elements $p_{ik} \geq 0$, $i \in J$, $k \in J$. For an arbitrary vector-function $\mathbf{x}$ with components $\{x_j(\cdot)\}$ defined on the interval $[0, T]$ introduce the class $\mathcal{Y}(T; \mathbf{x})$ o f vector-functions $\mathbf{y}$ with nonnegative monotone nondecreasing components $\{y_j(\cdot)\}$ on $[0, T]$, satisfying the system of inequalities:

$$y_j(t) \geq \sum_i y_i(t) p_{ij} - x_j(t), \quad \forall j \in J, \quad \forall t \in [0, T]. \tag{2.1}$$

Let $\mathbf{Y}(\cdot\,; \mathbf{x})$ denote the minimal element in the class $\mathcal{Y}(t; \mathbf{x})$:

$$\mathbf{Y}_j(t; \mathbf{x}) = \min\{y_j(t) : \mathbf{y} \in \mathcal{Y}(t; \mathbf{x})\}, \quad j \in J, \quad t \in [0, T], \tag{2.2}$$

if only it exists. Define the vector-function $\mathbf{Z}(\cdot\,; \mathbf{x})$ as

$$\mathbf{Z}_j(t; \mathbf{x}) = x_j(t) + \mathbf{Y}_j(t; \mathbf{x}) - \sum_i \mathbf{Y}_i(t; \mathbf{x}) p_{ij}, \quad j \in J, \quad t \in T. \tag{2.3}$$

Thus, the vector-functions $\mathbf{Y}(\cdot\,; \mathbf{x})$ and $\mathbf{Z}(\cdot\,; \mathbf{x})$ are non-anticipating mappings with respect to the second argument.

The following proposition is well-known (see, e.g. [2]).

Lemma 2.1. *If*

$$\mathbf{x} \in \mathcal{D}^J[0, T] \quad \text{and} \quad \mathcal{Y}(T; \mathbf{x}) \neq \emptyset, \tag{2.4}$$

then the minimal element $\mathbf{Y}(\cdot\,; \mathbf{x})$ *exists in the class* $\mathcal{Y}(T; \mathbf{x})$. *Moreover, for each* j*, the function* $Y_j(t; \mathbf{x})$ *increases only at those values of* $t \geq 0$ *when* $Z_j(t; \mathbf{x}) = 0$.

Introduce the matrix $H = H(P)$ with elements H_{ik} as

$$H = I + \sum_{n=1}^{\infty} P^n, \quad H = \{H_{ik} \in [0, \infty] : i \in J, \quad k \in J\}, \tag{2.5}$$

where I is the identity matrix. Later in the paper we use convention that

$$\delta_i H_{ij} = 0 \quad \text{if} \quad \delta_i = 0 \ \text{ and } \ H_{ij} = \infty. \tag{2.6}$$

By this agreement, the matrix H satisfies the equality

$$HP + I = H \tag{2.7}$$

even in the case when some elements of the matrix H are infinite.

The following proposition is very essential for our study.

Theorem 2.1. *Assume that a vector-function* $\mathbf{x}$ *satisfies (2.4) and a vector-function* $\mathbf{x}'$ *is such that*

$$\forall j \in J \quad \sum_i H_{ij} \|(x_i(\cdot) - x_i'(\cdot))^+\|_T < \infty.$$

Then the vector-function $\mathbf{Y}(\cdot\,;\mathbf{x}')$ *is defined on* $[0,T]$ *and the inequality*

$$Y_j(t,\mathbf{x}') \le Y_j(t,\mathbf{x}) + \sum_i H_{ij} \|(x_i(\cdot) - x_i'(\cdot))^+\|_t \tag{2.8}$$

holds for all $j \in J$ *and* $t \in [0,T]$.

Proof. We follow the proof of [9, Theorem 1]. Put

$$\underline{y}_j(t) = Y_j(t;\mathbf{x}), \quad \delta_j(t) = \|(x_j(\cdot) - x_j'(\cdot))^+\|_t,$$

$$y_j(t) = \underline{y}_j(t) + \sum_k \delta_k(t) H_{kj}, \quad j \in J, \quad t \in [0,T]. \tag{2.9}$$

By (2.1), we have $\underline{\mathbf{y}} \in \mathcal{Y}(T;\mathbf{x})$ and

$$\sum_i \underline{y}_i(t) p_{ij} \le \underline{y}_j(t) + x_j(t) \quad \forall j \in J, \quad \forall t \in [0,T]. \tag{2.10}$$

From (2.9), (2.10) and (2.7) it follows that

$$\sum_i y_i p_{ij} = \sum_i \underline{y}_i p_{ij} + \sum_{i,k} \delta_k(t) H_{kj} p_{ij} = \sum_i \underline{y}_i p_{ij} + \sum_i \delta_i(t) H_{ij} - \delta_j(t) \tag{2.11}$$

Now, from (2.9) and (2.11) we get

$$\sum_i y_i p_{ij} \le \underline{y}_j(t) + x_j(t) + \sum_i \delta_i(t) H_{ij} - (x_j(t) - x_j'(t)) =$$

$$= y_j(t) + x_j'(t), \quad \forall j \in J \quad \forall t \in [0,T].$$

From the last inequalities and the definition of the class $\mathcal{Y}(T,\mathbf{x}')$ we see that the function $\mathbf{y}(\cdot)$ defined by (2.9) belongs to this class. Hence, by (2.2), the minimal function $\mathbf{Y}(t,\mathbf{x}')$ does not exceed $\mathbf{y}(\cdot)$. The last statement can be written as inequalities (2.8).

Using (2.3) and (2.7), from Theorem 2.1 we obtain

Corollary 2.1. *If functions* $\mathbf{Y}(\cdot,\mathbf{x})$ *and* $\mathbf{Y}(\cdot,\mathbf{x}')$ *are defined, then*

$$\forall j \in J \quad \|Y_j(\cdot\,;\mathbf{x}) - Y_j(\cdot\,;\mathbf{x}')\|_t \le \sum_i H_{ij} \|x_i(\cdot) - x_i'(\cdot)\|_t,$$

$$\forall j \in J \quad \|Z_j(\cdot\,;\mathbf{x}) - Z_j(\cdot\,;\mathbf{x}')\|_t \le 2 \sum_i H_{ij} \|x_i(\cdot) - x_i'(\cdot)\|_t.$$

Remark 2.1. Let us give an useful interpretation for the mapping $\mathbf{Y}(\cdot\,;\mathbf{x})$ (see also [4]). Suppose we have a collection of buffers with infinite storage capacities, connected into a network, where a homogeneous fluid is circulating. Let function $x_j(t)$ be equal to the difference between total inflow to buffer j from the world outside the net and total outflow from buffer j to the world outside the net during time interval $[0,t]$. Assume a function $y_j(t)$ be the total inflow, which comes to buffer j from other buffers during time $[0,t]$. Suppose that if over a certain time buffer j gets some total inflow $h>0$ from others, then every buffer $i\in J$ loses flow $p_{ij}h$ for the same time; if a buffer gets some inflow from others, it is not possible to return this flow.

Under these assumptions, at moment t buffer j contains fluid flow, which equals

$$z_j(t)=x_j(t)+y_j(t)-\sum_i y_i(t)p_{ij},\quad j\in J.$$

In this interpretation buffer j needs to receive total internal flow $y_j(t)=Y_j(t;\mathbf{x})$ as the minimal one to guarantee its flow level $z_j(t)=Z_j(t;\mathbf{x})$ to be nonnegative.

Remark 2.2 In Sections 2, 3 and 4 we do not suppose that the matrix P is stochastic or substochastic. Thus inequality $\sum_i p_{ij}\le 1$ may be not fulfilled for some j.

3. Description of a deterministic queueing network

In this section we describe the example of the queueing network which we have in mind throughout the paper. The network consists of a finite or infinite set J of service stations indexed by $j\in J=\{1,2,...\}$. Customers are indistinguishable and they arrive at a station either exogenously or from other stations. Suppose the stations start serving the customers from the time $t=0$. Let $Q_j(0)$ be the number of customers initially present at station j, that is the initial length of queue j. Denote by $u_j(n)$ the time between the $n-1$th and nth arrival of exogenous customers to station j. Let $v_j(n)$ be the duration of the nth service performed at station j. Finally, assume that, the nth customer served at station j is routed to station k if $r_j(n)=k\in J$ and leaves the network if $r_j(n)\notin J$.

In Sections 3 and 4, all properties of the network are studied pathwise. For queueing networks with random components it corresponds to the assumption that we have fixed an elementary outcome from the appropriate probability space on which all random components of such network are defined. The dynamics of such network is completely defined by the following set of fixed numbers

$$\Big\{u_j(n),\,v_j(n)\in(0,\infty);\quad Q_j(0),\,r_j(n)\in\{0,1,2,\ldots\}\Big\},\quad j\in J,\quad n=1,2,\ldots. \tag{3.1}$$

For all $j \in J$ and $k \in J \cup \{0\}$ put

$$U_j(n) = \sum_{m=1}^{n} u_j(m), \quad V_j(n) = \sum_{m=1}^{n} v_j(m), \quad U_j(0) = V_j(0) = 0, \tag{3.2}$$

$$R_{jk}(n) = \sum_{m=1}^{n} I\{r_j(m) = k\}, \quad R_{jk}(0) = 0.$$

Here and in the following $I\{A\}$ denotes the indicator of the event A. Thus $R_{jk}(n)$ is the number of customers among the first n served at station j which are routed directly to station k.

For all $t \geq 0$ and $j \in J$ introduce functions $R_{jk}(t) = R_{jk}([t])$ and

$$A_{0j}(t) = \max\{n \geq 0 \,:\, U_j(n) \leq t\}, \quad S_j(t) = \max\{n \geq 0 \,:\, V_j(n) \leq t\}, \tag{3.3}$$

The function $A_{0j}(t)$ is the number of customers that arrived at station j from the world outside the net during the time interval $[0, t]$. The function $S_j(b)$ represents the number of customers served by server j during its first b units of busy-time.

Let $B_j(t)$ be the total amount of time that server j has been busy during the time interval $[0, t]$ and let $Q_j(t)$ be the number of customers present at station j at time t. Then the queue-length process $\mathbf{Q}(\cdot) = \{Q_j(t),\ t \geq 0,\ j \in J\}$ and the busy-time process $\mathbf{B}(\cdot) = \{B_j(t),\ t \geq 0,\ j \in J\}$ are bound by the flow-balance relation

$$Q_j(t) = Q_j(0) + A_{0j}(t) + \sum_i R_{ij}(S_i(B_i(t))) - S_j(B_j(t)). \tag{3.4}$$

Assume that the servers are serving the customers whenever their queues are not empty. Then the non-idling condition for server j has the form

$$B_j(t) = \int_0^t I\{Q_j(s) > 0\} ds, \quad j \in J,\ t \geq 0. \tag{3.5}$$

The representation (3.4) – (3.5) for the processes $\mathbf{B}(\cdot)$ and $\mathbf{Q}(\cdot)$ is due to J. M. Harrison. As a corollary we give one more useful representation. Choose arbitrary numbers $\mu_j > 0$ and $p_{ij} \geq 0$, where $i,\ j \in J$, and define

$$Y_j(t) = \mu_j(t - B_j(t)), \quad X_j(t) = Q_j(t) - Y_j(t) + \sum_i Y_i(t) p_{ij}. \tag{3.6}$$

The following proposition is well known (see, e.g. [3, p.432]).

Lemma 3.1 *For the vector-functions* $\mathbf{X}(\cdot)$ *and* $\mathbf{Y}(\cdot)$ *defined in (3.6) the following equalities hold*

$$\mathbf{Y}(t) = \mathbf{Y}(t; \mathbf{X}), \quad \mathbf{Q}(t) = \mathbf{Z}(t; \mathbf{X}),$$

where the mappings $\mathbf{Y}(\cdot\,;\cdot)$ *and* $\mathbf{Z}(\cdot\,;\cdot)$ *were introduced in Section 2.*

The proof bases on the fact that the functions $\{Y_j(\cdot)\}$ and functions

$$Z_j(\cdot) = X_j(\cdot) + Y_j(\cdot) - \sum_i Y_i(\cdot)p_{ij}, \quad j \in J,$$

satisfy the equality

$$\mu_j(t - B_j(t)) = Y_j(t) = \mu_j \int\limits_0^t I\{Z_j(s) = 0\}ds. \tag{3.7}$$

This equality follows from (3.5), as by definition (3.6) of $\mathbf{X}(\cdot)$ we have $\mathbf{Q}(\cdot) \equiv \mathbf{Z}(\cdot)$. From (3.7) it is easily seen that the function $Y_j(\cdot)$ increases only at those times $t \geq 0$ when $Z_j(t) = 0$. Hence, Lemma 3.1 follows from Lemma 2.1.

Remark 3.1. For all j put

$$\overline{X}_j(t;\mathbf{B}) = Q_j(0) + A_{0j}(t) + \sum_i R_{ij}(S_i(B_i(t))) -$$

$$-S_j(B_j(t)) - \mu_j(t - B_j(t)) + \sum_i \mu_i(t - B_i(t))p_{ij},$$

$$\overline{B}_j(t;\mathbf{X}) = t - Y_j(t;\mathbf{X})/\mu_j, \quad \overline{Q}_j(t;\mathbf{X}) = Z_j(t;\mathbf{X}).$$

So we have now given the rigorous definitions for the mappings we had used in the introduction.

4. The key results

Fix the numbers $\mu_j > 0, \quad p_{ij} \geq 0, \quad i, j \in J,$ and put

$$\tilde{\mu}_j = \sum_i \mu_i p_{ij} - \mu_j, \quad \tilde{S}_j(t) = S_j(t) - \mu_j t, \quad \tilde{R}_{ij}(t) = R_{ij}(t) - p_{ij}t, \tag{4.1}$$

$$\tilde{F}_{ij}(t) = R_{ij}(S_i(t)) - \mu_i p_{ij} t = \tilde{R}_{ij}(\mu_i t + \tilde{S}_i(t)) + \tilde{S}_i(t)p_{ij}. \tag{4.2}$$

In this notation Lemma 3.1 can be rewritten as

$$\mu_j(t - B_j(t)) = Y_j(t) = Y_j(t;\mathbf{X}) \qquad Q_j(t) = Z_j(t;\mathbf{X}), \tag{4.3}$$

with

$$X_j(t) = \tilde{\mu}_j t + Q_j(0) + A_{0j}(t) - \tilde{S}_j(B_j(t)) + \sum_i \tilde{F}_{ij}(B_i(t)) \tag{4.4}$$

for all $j \in J$ and $t \geq 0$.

Now we describe the three-step procedure which allows to obtain approximations of the functions $\mathbf{B}(\cdot)$ and $\mathbf{Q}(\cdot)$ simultaneously and gives explicit estimates of remainder terms in Theorems 4.1 – 4.3.

In the first step we have to choose a function $\hat{\mathbf{X}}_0(\cdot)$ with the components

$$\hat{X}_{0j}(t) = \tilde{\mu}_j t + a_j(t), \quad j \in J, \tag{4.5}$$

as a simple initial approximation for the function $\mathbf{X}(\cdot)$ and take $\{b_j(\cdot)\}$ as solutions of the equations

$$\mu_j(t - b_j(t)) = Y_j(t; \hat{\mathbf{X}}_0), \quad j \in J. \tag{4.6}$$

Here $a_j(\cdot)$ is interpreted as some approximation for the function $Q_j(0) + A_{0j}(\cdot)$ and $b_j(\cdot)$ is used as an initial approximation for $B_j(\cdot)$.

To estimate the remainder term arising at the first step, we need the following notation. Put

$$\alpha_{0j}(t) = \|Q_j(0) - A_{0j}(\cdot) - a_j(\cdot)\|_t + \|\tilde{S}_i(\cdot)\|_t, \tag{4.7}$$

$$\alpha_{ij}(t) = \|\tilde{F}_{ij}(\cdot)\|_t, \quad \alpha_j(t) = \Big(\sum_i \alpha_{0i}(t) H_{ij} + \sum_{i,k} \alpha_{ik}(t) H_{kj}\Big)/\mu_j.$$

Remark 4.1. By definitions (4.1) and (4.2),

$$\alpha_{ij}(t) \leq \|\tilde{R}_{ij}(\cdot)\|_{S_i(t)} + p_{ij}\|\tilde{S}_i(\cdot)\|_t. \tag{4.8}$$

Theorem 4.1. *For all $j \in J$ and $t \geq 0$,*

$$\|B_j(\cdot) - b_j(\cdot)\|_t \leq \alpha_j(t). \tag{4.9}$$

This proposition follows immediately from (4.3) – (4.7) and Corollary 2.1 with $\mathbf{x} = \mathbf{X}$ and $\mathbf{x}' = \hat{\mathbf{X}}_0$, as

$$\|X_j(\cdot) - \hat{X}_{0j}(\cdot)\|_t \leq \alpha_{0j}(t) + \sum_i \alpha_{ij}(t).$$

Remark 4.2. The choice of $\hat{X}_{0j}(\cdot)$ as

$$\hat{X}_{0j}(t) = \tilde{\mu}_j t + Q_j(0) + A_{0j}(t), \quad \forall j \in J, \quad \forall t \geq 0,$$

results in the maximal accuracy in Theorem 4.1. However, this makes the functions $b_j(\cdot)$ too complex, and therefore, the second-step functions become hard to use.

Remark 4.3. We could have chosen $\hat{X}_{0j}(\cdot)$ as

$$\hat{X}_{0j}(t) = \tilde{\mu}_j t + Q_j(0) + a_{0j}(t). \quad \forall j \in J, \quad \forall t \geq 0,$$

where $a_{0j}(\cdot)$ is some approximation for function $A_{0j}(\cdot)$. In this case $a_{0j}(\cdot)$ do not depends on $Q_j(0)$. The traditional way (see [12, 4]) is to choose functions $\hat{X}_{0j}(\cdot)$ as

$$\hat{X}_{0j}(t) = \tilde{\mu}_j t + a_{0j}(t), \quad \forall j \in J, \quad \forall t \geq 0.$$

We emphasize that in this case the functions $\{a_{0j}(\cdot)\}$ should not necessarily have the linear form $a_{0j}(t) = \lambda_{0j} t$. Therefore, Theorems 4.1 and 4.2 may be applied for more general processes than those considered in Section 5 and in [2, 3, 4, 5, 11, 12].

At the second step we approximate the function $X_j(\cdot)$ by

$$\hat{X}_j(t) = \tilde{\mu}_j t + \hat{Q}_j(0) + \hat{A}_{0j}(t) - \hat{S}_j(b_j(t)) + \sum_i \hat{F}_{ij}(b_i(t)), \tag{4.10}$$

where $\hat{Q}_j(0)$, $\hat{A}_{0j}(\cdot)$, $\hat{S}_j(\cdot)$, and $\hat{F}_{ij}(\cdot)$ have to be chosen as approximations for $Q_j(0)$, $A_{0j}(\cdot)$, $\tilde{S}_j(\cdot)$, and $\tilde{F}_{ij}(\cdot)$. Introduce the functions

$$\hat{\mathbf{Y}}(t) = \mathbf{Y}(t; \hat{\mathbf{X}}), \quad \hat{\mathbf{Q}}(t) = \mathbf{Z}(t; \hat{\mathbf{X}}) \tag{4.11}$$

as approximations for $\mathbf{Y}(\cdot)$ and $\mathbf{Q}(\cdot)$.

To estimate the remainder term arising at the second step, put

$$\begin{aligned}\Delta_{0j}(t; \hat{\mathbf{X}}) &= \|Q_j(0) + A_{0j}(\cdot) - \hat{Q}_j(0) - \hat{A}_{0j}(\cdot)\|_t + \\ &+ \|\tilde{S}_j(\cdot) - \hat{S}_j(\cdot)\|_t + \beta(\alpha_j(t); t, \hat{S}_j(\cdot)), \\ \Delta_{ij}(t; \hat{\mathbf{X}}) &= \|\tilde{F}_{ij}(\cdot) - \hat{F}_{ij}(\cdot)\|_t + \beta(\alpha_i(t); t, \hat{F}_{ij}(\cdot)), \\ \Delta_j(t; \hat{\mathbf{X}}) &= \sum_i \Delta_{0i}(t; \hat{\mathbf{X}}) H_{ij} + \sum_{i,k} \Delta_{ik}(t; \hat{\mathbf{X}}) H_{kj}.\end{aligned} \tag{4.12}$$

The functions $\beta(\cdot\,; \cdot\,, \cdot)$ were defined in (1.3).

Theorem 4.2. *For all $j \in J$ and $t \geq 0$,*

$$\|X_j(\cdot) - \hat{X}_j(\cdot)\|_t \leq \Delta_{0j}(t; \hat{\mathbf{X}}) + \sum_i \Delta_{ij}(t; \hat{\mathbf{X}}), \tag{4.13}$$

$$\|Y_j(\cdot) - \hat{Y}_j(t)\|_t \leq \Delta_j(t; \hat{\mathbf{X}}), \quad \|Q_j(\cdot) - \hat{Q}_j(t)\|_t \leq 2\Delta_j(t; \hat{\mathbf{X}}). \tag{4.14}$$

Proof. Note that, for $s \in [0, t]$,

$$|\tilde{S}_j(B_j(s)) - \hat{S}_j(b_j(s))| \leq |\tilde{S}_j(B_j(s)) - \hat{S}_j(B_j(s))| +$$

$$+|\hat{S}_j(B_j(s)) - \hat{S}_j(b_j(s))| \le \|\tilde{S}_j(\cdot) - \hat{S}_j(\cdot)\|_t + \beta(\alpha_j(t); t, \hat{S}_j(\cdot)). \tag{4.15}$$

We use definition (1.3) and estimate (4.9) to obtain the last inequality in (4.15). Similarly,

$$|\tilde{F}_{ij}(B_i(s)) - \hat{F}_{ij}(b_i(s))| \le |\tilde{F}_{ij}(B_i(s)) - \hat{F}_{ij}(B_i(s))| +$$

$$+|\hat{F}_{ij}(B_i(s)) - \hat{F}_{ij}(b_i(s))| \le \Delta_{ij}(t; \hat{\mathbf{X}}). \tag{4.16}$$

Now (4.13) follows from inequalities (4.15), (4.16) and definitions (4.4), (4.10) and (4.12). Inequalities (4.14) are a consequence of (4.13) and Corollary 2.1 with $\mathbf{x} = \mathbf{X}$ and $\mathbf{x}' = \hat{\mathbf{X}}$.

Remark 4.4. The maximal accuracy in Theorem 4.2 is obtained with

$$\hat{Q}_j(0) = Q_j(0), \quad \hat{A}_{0j}(\cdot) = A_{0j}(\cdot), \quad \hat{S}_j(\cdot) = \tilde{S}_j(\cdot),$$

$$\hat{F}_{ij}(\cdot) = \tilde{F}_{ij}(\cdot), \quad \forall i \in J, \quad \forall j \in J.$$

In this case

$$\Delta_{0j}(t; \hat{\mathbf{X}}) = \beta(\alpha_j(t); t, \tilde{S}(\cdot)), \quad \Delta_{ij}(t; \hat{\mathbf{X}}) = \beta(\alpha_i(t); t, \tilde{F}_{ij}(\cdot)).$$

To proceed to the third step of approximation, assume that the two families of random processes $Q_j(0) + A_{0j}(\cdot)$, $\tilde{S}_j(\cdot)$, $\tilde{F}_{ij}(\cdot)$ and $\zeta_j(0) + \xi_{0j}(\cdot)$, $\tilde{\xi}_j(\cdot)$, $\tilde{\xi}_{ij}(\cdot)$ are defined on the same probability space. Introduce a random process $\hat{\boldsymbol{\xi}}$ with the components

$$\hat{\xi}_j(t) = \tilde{\mu}_j t + \zeta_j(0) + \xi_{0j}(t) - \tilde{\xi}_j(b_j(t)) + \sum_i \tilde{\xi}_{ij}(b_i(t)) \tag{4.17}$$

as an approximation for $\hat{\mathbf{X}}$ and the processes

$$\boldsymbol{\eta}(t) = \mathbf{Y}(t; \hat{\boldsymbol{\xi}}) \qquad \boldsymbol{\zeta}(t) = \mathbf{Z}(t; \hat{\boldsymbol{\xi}}) \tag{4.18}$$

as approximations for the process $\mathbf{Y}(\cdot)$ and the queue-length process $\mathbf{Q}(\cdot)$ respectively.

Now we estimate the remainder ter ms arising at the third step.

Theorem 4.3. *For all $j \in J$ and $t \ge 0$*

$$\|X_j(\cdot) - \hat{\xi}_j(\cdot)\|_t \le \Delta_{0j}(t; \hat{\boldsymbol{\xi}}) + \sum_i \Delta_{ij}(t; \hat{\boldsymbol{\xi}}),$$

$$\|Y_j(\cdot) - \eta_j(\cdot)\|_t \le \Delta_j(t; \hat{\boldsymbol{\xi}}), \quad \|Q_j(\cdot) - \zeta_j(\cdot)\|_t \le 2\Delta_j(t; \hat{\boldsymbol{\xi}}).$$

Formally, this proposition is a special case of Theorem 4.2 with $\hat{Q}_j(0) = \zeta_j(0)$, $\hat{A}_{0j}(\cdot) = \xi_{0j}(\cdot)$, $\hat{S}_j(\cdot) = \tilde{\xi}_j(\cdot)$, and $\hat{F}_{ij}(\cdot) = \tilde{\xi}_{ij}(\cdot)$.

Remark 4.5. If $A_{0j}(\cdot)$, $\tilde{S}_j(\cdot)$, and $\tilde{F}_{ij}(\cdot)$ are regarded as random processes, Theorems 4.1 and 4.2 hold *pathwise.* We will have similar situation in Theorem 4.3, if we treat functions $\xi_{0j}(\cdot)$, $\tilde{\xi}_j(\cdot)$, and $\tilde{\xi}_{ij}(\cdot)$ as specially chosen ones with some desirable properties. The fact that the considered objects are random will become essential later, in the applications.

Remark 4.6. At the third step it is natural to choose $\xi_{0j}(\cdot)$, $\tilde{\xi}_j(\cdot)$, and $\tilde{\xi}_{ij}(\cdot)$ as random processes with "good-behaving" distributions. If these distributions are close, in some sense, to those of the processes $A_{0j}(\cdot)$, $\tilde{S}_j(\cdot)$, and $\tilde{F}_{ij}(\cdot)$, we may apply the ideas of A. V. Skorohod [14] and construct these processes on a single probability space in such a way that the remainder terms $\Delta_{\bullet}(\cdot\,;\hat{\boldsymbol{\xi}})$ in Theorem 4.3 would be small. An important example of when it is possible is considered in Section 5. Another example is given in Remark 4.8.

Remark 4.7. At the third step, the processes $\hat{\mathbf{X}}$ and $\hat{\boldsymbol{\xi}}$ introduced in (4.10) and (4.17) may be defined on a single probability space without paying attention to the behavior of the summands forming these processes. In this case the useful estimates

$$\|Y_j(\cdot)-\eta_j(\cdot)\|_t \le \Delta_j(t;\hat{\mathbf{X}}) + \sum_i \|\hat{X}_i(\cdot)-\hat{\xi}_i(\cdot)\|_t H_{ij},$$

$$\|Q_j(\cdot)-\zeta_j(\cdot)\|_t \le 2\Delta_j(t;\hat{\mathbf{X}}) + 2\sum_i \|\hat{X}_i(\cdot)-\hat{\xi}_i(\cdot)\|_t H_{ij}$$

follow from Corollary 2.1 and Theorem 4.2.

Remark 4.8. Let the set J be finite. Assume that for all $T>0$ we have the random process $\mathbf{X}^{(T)}(\cdot)$ defined on the interval $[0,T]$ with a sample path from the space $\mathcal{D}^J[0,T]$. Denote by

$$\overline{\mathbf{X}}^{(T)}(t) = \pi^{-1}(T)\hat{\mathbf{X}}^{(T)}(tT), \quad t\in[0,1]$$

the normalization of this process by some function $\pi(T)>0$. Suppose that

$$\overline{\mathbf{X}}^{(T)}(\cdot) \Rightarrow_d \bar{\boldsymbol{\xi}}(\cdot), \tag{4.19}$$

where $\bar{\boldsymbol{\xi}}$ is a process on $[0,1]$ with continuous paths and $\Rightarrow_d$ denotes the C-convergence (see [1]). This is equivalent to convergence in distribution $f(\overline{\mathbf{X}}^{(T)}) \Rightarrow f(\bar{\boldsymbol{\xi}})$ as $T\to\infty$, holding for all functionals f on $\mathcal{D}^J[0,T]$, continuous with respect to the uniform norm.

The Strassen theorem [15] and the remark of A. V. Skorohod [14] allow us to enlarge the probability space on which the process $\overline{\mathbf{X}}^{(T)}$ is defined and construct a random process $\bar{\boldsymbol{\xi}}^{(T)}$, so that it is identically distributed as $\bar{\boldsymbol{\xi}}$ and, moreover,

$$\forall j\in J \quad \|\overline{X}_j^{(T)}(\cdot)-\bar{\xi}_j^{(T)}(\cdot)\|_1 \to_p 0, \quad \text{as} \quad T\to\infty. \tag{4.20}$$

Relation (4.20) can also be written as

$$\forall j \in J \quad \|\hat{X}_j^{(T)}(\cdot) - \hat{\xi}_j^{(T)}(\cdot)\|_T =_p o(\pi(T)), \quad \text{where} \quad \hat{\boldsymbol{\xi}}^{(T)}(t) = \pi(t)\overline{\boldsymbol{\xi}}^{(T)}(t/T). \tag{4.21}$$

We see that convergence (4.19) allows to obtain a representation on a single probability space with the rates of convergence (4.20) and (4.21). Hence, the assumption of Theorem 4.3 on the processes $\hat{\mathbf{X}}$ and $\hat{\boldsymbol{\xi}}$ being defined on the same probability space, is not very restrictive.

Remark 4.9. In the next section it is convenient to use the following representation for the processes $\tilde{\xi}_{ij}(\cdot)$

$$\tilde{\xi}_{ij}(t) = \breve{\xi}_{ij}(\mu_i t) + \tilde{\xi}_i(t)p_{ij}, \tag{4.22}$$

where $\breve{\xi}_{ij}(\cdot)$ are regarded as approximations for $\tilde{R}_{ij}(\cdot)$. In this case we may use the estimate

$$\Delta_{ij}(t;\hat{\boldsymbol{\xi}}) \leq \|\tilde{R}_{ij}(\cdot) - \breve{\xi}_{ij}(\cdot)\|_{S_i(t)} + \beta\Big(\|\tilde{S}_i(\cdot)\|_t, \mu_i t, \breve{\xi}_{ij}(\cdot)\Big) +$$

$$+ \|\tilde{S}_i(\cdot) - \tilde{\xi}_i(\cdot)\|_t p_{ij} + \beta(\mu_i\alpha_i(t); \mu_i t, \breve{\xi}_{ij}(\cdot)) + \beta(\alpha_i(t); t, \tilde{\xi}_i(\cdot))p_{ij}. \tag{4.23}$$

which follows from definitions (4.2), (4.22), and (4.12) with $\hat{F}_{ij}(\cdot) \equiv \tilde{\xi}_{ij}(\cdot)$.

Remark 4.10 All the results of this section hold for any queueing network for which Lemma 3.1 is valid. This class of networks is wider than the class described in Section 3. For instance, it includes all queueing networks satisfying Theorem 2.1 in [3].

5. Accuracy of approximations for open queueing networks

In this section we assume that the components of the queueing network described in Section 3 are random and depend on some parameter $T > 0$. In what follows, we preserve the notation from Sections 3 and 4, adding the superscript (T) where necessary.

For all $T > 0$ and $j \in J$ let $\{\tilde{u}_j^{(T)}(\cdot)\}$, $\{\tilde{v}_j^{(T)}(\cdot)\}$, $\{r_j^{(T)}(\cdot)\}$ be mutually independent sequences of i. i. d. random variables. Let $\mathbf{Q}^{(T)}(0) = (Q_1^{(T)}(0), Q_2^{(T)}(0), \dots)$. be a random vector, independent of them. Suppose that the inter-arrival times $\{u_j^{(T)}(n)\}$ and service times $\{v_j^{(T)}(n)\}$ have the form

$$v_j^{(T)}(n) = (1 + \tilde{v}_j^{(T)}(n))/\mu_j(T) \geq 0, \quad \text{for all} \quad j \in J, \tag{5.1}$$

$$u_j^{(T)}(n) = (1 + \tilde{u}_j^{(T)}(n))/\lambda_{0j}(T) \geq 0 \quad \text{if} \quad \lambda_{0j}(T) > 0, \tag{5.2}$$

where

$$0 \leq \lambda_{0j}(T) < \infty, \quad 0 < \mu_j(T) < \infty \quad \forall j \in J \tag{5.3}$$

are some numbers. Assume that the distributions of routing indicators $r_j^{(T)}(\cdot)$ do not depend on T and that the elements of the routing matrix P are

$$p_{ik} = \mathbf{P}(r_i^{(T)} = k), \quad k \in J \cup \{0\}. \tag{5.4}$$

We suppose below that this matrix P coincides with the matrix P used in definitions in Sections 2 and 3. Finally, assume for simplicity that

$$J = \{1, 2, ..., |J|\}, \quad |J| < \infty, \quad \max_{i,k \in J} H_{ik} < \infty, \tag{5.5}$$

where the matrix H with elements H_{ik} was defined in (2.5).

Assumptions (5.1) – (5.5) mean that now we consider open queueing network with finite set of stations and fixed routing matrix P with a spectral radius strictly less than 1. But the distributions of the inter-arrival and service times may essentially depend on $T \to \infty$.

Let $\lambda_j(T),\ j \in J$, denote the vector with components being the maximal solutions of the traffic equations

$$\lambda_j(T) = \lambda_{0j}(T) + \sum_i \min\{\lambda_i(T), \mu_i(T)\} p_{ij}, \quad j \in J. \tag{5.6}$$

The vector $\{\lambda_j(T)\}$ always exists (see, e.g. [2, Theorem 3.1]).

Now take

$$b_j(t) = \rho_j(T)t, \quad \text{where} \quad \rho_j(T) = \min\{1, \lambda_j(T)/\mu_j(T)\} \tag{5.7}$$

as the first approximation for the busy-time process $B_j(\cdot)$. This corresponds to

$$\hat{X}_{0j}(t) = a_{0j}(t) \equiv \lambda_{0j}(T)t \tag{5.8}$$

in (4.5) and (4.6).

Introduce $\hat{\mathbf{X}}^{(T)}(\cdot)$ with the components

$$\hat{X}_j^{(T)}(t) = \tilde{\mu}_j(T)t + Q_j^{(T)}(0) + A_{0j}^{(T)}(t) - \tilde{S}_j^{(T)}(\rho_j(T)t) +$$
$$+ \sum_i \tilde{S}_i^{(T)}(\rho_i(T)t)p_{ij} + \sum_i \tilde{R}_{ij}^{(T)}(\mu_i(T)\rho_i(T)t). \tag{5.9}$$

For the second approximation of the busy-time and queue-length processes take

$$\hat{\mathbf{Y}}^{(T)}(t) = \mathbf{Y}(t; \hat{X}^{(T)}), \quad \hat{\mathbf{Q}}^{(T)}(t) = \mathbf{Z}(t; \hat{X}^{(T)}). \tag{5.10}$$

Let us investigate the accuracy of the approximations in (5.6) – (5.10) when

$$\forall j \in J \quad \forall T > 0 \quad \mathbf{E}\tilde{u}_j^{(T)}(1) = \mathbf{E}\tilde{v}_j^{(T)}(1) = 0,$$

$$\sigma_{0j}^2(T) = \mathbf{E}(\tilde{u}_j^{(T)}(1))^2 < \infty, \quad \sigma_j^2(T) = \mathbf{E}(\tilde{v}_j^{(T)}(1))^2 < \infty.$$

By convention,

$$A_{0j}^{(T)}(\cdot) \equiv \sigma_{0j}(T) = 0 \quad \text{if} \quad \lambda_{0j}(T) = 0. \tag{5.11}$$

This trivial case will be considered separately; the results for it follow immediately from relations (5.2) and (5.11).

Remark 5.1. From now on $T \to \infty$ in the limits. Thus, the notation

$$a(T) = O(b(T)) \quad \text{or} \quad a(T) = o(b(T))$$

means, as usual, that

$$\exists T_0 < \infty \quad \exists c < \infty : |a(T)| \le c|b(T)| \quad \text{for all} \quad |T| \ge T_0,$$

or respectively

$$\forall \varepsilon > 0 \quad \exists T_0(\varepsilon) < \infty : |a(T)| \le \varepsilon|b(T)| \quad \text{for all} \quad |T| \ge T_0(\varepsilon).$$

Similarly, the notation

$$a(T) =_p O(b(T)) \quad \text{or} \quad a(T) =_p o(b(T))$$

means that

$$\limsup_{T\to\infty} \mathbf{P}(|a(T)| \ge x|b(T)|) \to 0 \quad \text{as} \quad x \to \infty,$$

or respectively

$$\forall \varepsilon > 0 \quad \lim_{T\to\infty} \mathbf{P}(|a(T)| \ge \varepsilon|b(T)|) = 0.$$

Theorem 5.1. *Let $r_0(T) > 0$ be such that*

$$\forall j \in J \quad Q_j^{(T)}(0) =_p O(r_0(T)), \quad T\mu_j(T) = O(r_0^2(T)),$$

$$T\lambda_{0j}(T)\sigma_{0j}^2(T) + T\mu_j\sigma_j^2(T) = O(r_0^2(T)) \tag{5.12}$$

and

$$\forall j \in J \quad \sigma_{0j}^2(T) = o(T\lambda_{0j}(T)), \quad \sigma_j^2(T) = o(T\mu_j(T)). \tag{5.13}$$

Then for all j

$$\mu_j(T) \sup_{t \le T} |B_j^{(T)}(t) - \rho_j(T)t| =_p O(r_0(T)). \tag{5.14}$$

The maximal accuracy at the first step is obtained if Theorem 5.1 holds with $r_0(T) = T^{1/2}$.

Let us investigate the question of accuracy at the second step of approximation.

Theorem 5.2. *Assume that for some numbers $r(T) > 0$ and $s > 2$, the following two conditions hold*

$$\forall j \in J \quad T\lambda_{0j}(T)\mathbf{E}\min\left\{\frac{|\tilde{u}_j^{(T)}(1)|^s}{r^s(T)}, \frac{|\tilde{u}_j^{(T)}(1)|^2}{r^2(T)}\right\} \to 0, \tag{5.15}$$

$$\forall j \in J \quad T\mu_j(T)\mathbf{E}\min\left\{\frac{|\tilde{v}_j^{(T)}(1)|^s}{r^s(T)}, \frac{|\tilde{v}_j^{(T)}(1)|^2}{r^2(T)}\right\} \to 0. \tag{5.16}$$

Let the conditions of Theorem 5.1 be satisfied and

$$r_0(T) \geq r(T) \to \infty, \quad \ln(1 + T\mu_j(T)) = O(r(T)),$$

$$(1 + \sigma_j^2(T))r_0(T)\ln(1 + T\mu_j(T)/r_0(T)) = O(r^2(T)), \tag{5.17}$$

$$\sigma_{0j}^2(T)\ln(1 + T\lambda_{0j}(T)/r(T)) + \sigma_j^2(T)\ln(1 + T\mu_j(T)/r(T)) = O(r(T))$$

for all j. Then for each j

$$\|X_j^{(T)}(\cdot) - \hat{X}_j^{(T)}(\cdot)\|_T =_p O(r(T)),$$

$$\|Y_j^{(T)}(\cdot) - \hat{Y}_j^{(T)}(\cdot)\|_T =_p O(r(T)), \quad \|Q_j^{(T)}(\cdot) - \hat{Q}_j^{(T)}(\cdot)\|_T =_p O(r(T)).$$

Corollary 5.1. *Suppose that for all $j \in J$ the Lindeberg conditions*

$$\forall \varepsilon > 0 \quad T\lambda_{0j}(T)r^{-2}(T)\mathbf{E}\{(\tilde{u}_j^{(T)}(1))^2;\ |\tilde{u}_j^{(T)}(1)| > \varepsilon r(T)\} \to 0,$$

$$\forall \varepsilon > 0 \quad T\mu_j(T)r^{-2}(T)\mathbf{E}\{(\tilde{v}_j^{(T)}(1))^2;\ |\tilde{v}_j^{(T)}(1)| > \varepsilon r(T)\} \to 0$$

are satisfied. Assume the conditions of Theorem 5.1 are satisfied with

$$r_0(T) = T^{1/2+c} \quad \text{for some} \quad c \in [0, 1/2).$$

Let also

$$\forall j \in J \qquad \sigma_{0j}^2(T) + T^c\sigma_j^2(T) = o(T^{1/2}/\ln T).$$

Then Theorem 5.2 holds with

$$r(T) = o(T^{1/2}).$$

Now we establish the results for the accuracy of approximation at the third step. Assume that the probability space, on which the random variables $\{u_\bullet^{(T)}(\cdot), v_\bullet^{(T)}(\cdot), r_\bullet^{(T)}(\cdot), Q_\bullet^{(T)}(0)\}$ are defined, is rich enough, and that the conditions of Theorem 5.2 are satisfied. In this case, by Lemmas 5.4, 5.5, and 5.6 which will be proved below, for all $T > 0$ we may construct mutually independent Wiener processes $\{W_{0j}^{(T)}(\cdot), W_j^{(T)}(\cdot), W_{jk}^{(T)}(\cdot), j \in J, k \in J \cup \{0\}\}$, so that the following relations hold

$$\gamma_{0j}(T) \equiv \sup_{t \le T} |A_{0j}^{(T)}(t) - \lambda_{0j}(T)t - \lambda_{0j}^{1/2}(T)\sigma_{0j}(T)W_{0j}^{(T)}(t)| =_p O(r(T)), \quad (5.18)$$

$$\gamma_j(T) \equiv \sup_{t \le T} |S_j^{(T)}(t) - \mu_j(T)t - \mu_j^{1/2}(T)\sigma_j(T)W_j^{(T)}(t)| =_p O(r(T)), \quad (5.19)$$

$$\gamma_{jk}(T) = \sup_{t \le 2T\mu_j(T)} |R_{jk}^{(T)}(t) - p_{jk}t - p_{jk}^{1/2}W_{jk}^{(T)}(t) + p_{jk}\overline{W}_j^{(T)}(t)| =_p O(r(T)), \quad (5.20)$$

where

$$\overline{W}_j^{(T)}(t)| = \sum_{k \in J \cup \{0\}} p_{jk}^{1/2} W_{jk}^{(T)}(t). \quad (5.21)$$

Hereinafter we refer to the Brownian motion $W(t)$, such that $\mathbf{E}W(t) \equiv 0$ and $\mathbf{E}W^2(t) \equiv t$, as a Wiener process.

For $j \in J$, put

$$\hat{\xi}_j^{(T)}(t) = Q_j^{(T)}(0) + \lambda_j(T)t - \mu_j(T)t + \lambda_{0j}^{1/2}(T)\sigma_{0j}(T)W_{0j}^{(T)}(t) -$$
$$-\mu_j^{1/2}(T)\sigma_j(T)W_j^{(T)}(\rho_j(T)t) + \sum_i p_{ij}\mu_i^{1/2}(T)\sigma_i(T)W_i^{(T)}(\rho_i(T)t) + \quad (5.22)$$
$$+ \sum_i p_{ij}^{1/2} W_{ij}(\mu_i(T)\rho_i(T)t) - \sum_i p_{ij}\overline{W}_i(\mu_i(T)\rho_i(T)t).$$

As in (4.17) and (4.18) introduce

$$\hat{\boldsymbol{\eta}}^{(T)}(t) = \mathbf{Y}(t; \hat{\boldsymbol{\xi}}^{(T)}), \quad \hat{\boldsymbol{\zeta}}^{(T)}(t) = \mathbf{Z}(t; \hat{\boldsymbol{\xi}}^{(T)}).$$

Theorem 5.3. *Let all conditions of Theorem 5.2 be satisfied. Then for all $T > 0$ the Wiener processes $W_{0j}^{(T)}(\cdot)$, $W_j^{(T)}(\cdot)$, $W_{jk}^{(T)}(\cdot)$, $j \in J$, $k \in J \cup \{0\}$, may be defined in such a way that they are mutually independent, independent of $\{Q_j(0); j \in J\}$, and, moreover, the following relations hold*

$$\forall j \in J \quad \|X_j^{(T)}(\cdot) - \hat{\xi}_j^{(T)}(\cdot)\|_T = O(r(T)),$$

$$\|Y_j^{(T)}(\cdot) - \zeta_j^{(T)}(\cdot)\|_T = O(r(T)), \quad \|Q_j^{(T)}(\cdot) - \eta_j^{(T)}(\cdot)\|_T = O(r(T)).$$

Corollary 5.2. *If the conditions of Corollary 5.1 are satisfied then Theorem 5.3 holds with* $r(T) = o(T^{1/2})$.

We obtain the maxima l accuracy at the second and the third steps in the following case.

Corollary 5.3. *Assume that for some numbers* $s \geq 4$, $\gamma \geq 0$ *and* $\delta \geq 0$ *the following conditions hold for all* $j \in J$

$$\lambda_{0j}(T)\mathbf{E}|\tilde{u}_j^{(T)}(1)|^s + \mu_j(T)\mathbf{E}|\tilde{v}_j^{(T)}(1)|^s = o(T^{(s-4)/4}(\ln T)^{(1/2+\gamma+\delta)s}),$$

$$\sigma_{0j}^2(T) = o(T^{1/2}/\ln T), \quad \sigma_j^2(T) = o(\ln T)^{2\delta}.$$

Assume that the conditions of Theorem 5.1 are satisfied with

$$r_0(T) = T^{1/2}(\ln T)^{2\gamma}.$$

Then Theorem 5.2 and Theorem 5.3 hold with

$$r(T) = T^{1/4}(\ln T)^{1/2+\gamma+\delta}.$$

Remark 5.2. H. Chen and A. Mandelbaum [4] investigated the case when the random variables $\tilde{u}_j^{(T)}(\cdot)$ and $\tilde{v}_j^{(T)}(\cdot)$ introduced in (5.1) do not depend on T. When, in addition, these variables have exponential moments, L. Horvath [11] obtained for one- or two-station network the estimates that are stronger than those following from Corollary 5.3. For the simpler case, when the random variables $u_j^{(T)}(\cdot)$ and $v_j^{(T)}(\cdot)$ introduced in (5.1) do not depend on T, the unimprovable rate of convergence was obtained by H. Chen and A. Mandelbaum in [5]. From the point of view of convergence in probability Corollary 5.3 gives the analogous accuracy of approximations as it is possible to derive from [5, 11].

Remark 5.3. It may be shown that Corollary 5.2 holds under weaker conditions than the theorems of diffusion approximation in [12, 4]. For example, we were not assuming that

$$\sigma_{0j}(T) + \sigma_j(T) = O(1).$$

On the other hand, from Remark 4.8 with $\pi(T) = T^{1/2}$ it follows that Corollary 5.2 gives the same accuracy as these theorems. This effect is due to a wider class of processes $\hat{\boldsymbol{\xi}}^{(T)}(\cdot)$ that we used for approximation of the process $\mathbf{X}^{(T)}(\cdot)$, as we allow the distribution of the process $T^{-1/2}\hat{\boldsymbol{\xi}}^{(T)}(tT)$ to depend on T. In other words, in Theorem 5.3 and its corollaries we have come from the approximation of normalized processes

$$\overline{\mathbf{X}}^{(T)}(t) = T^{-1/2}\mathbf{X}^{(T)}(tT), \quad \overline{\mathbf{Y}}^{(T)}(t) = T^{-1/2}\mathbf{Y}^{(T)}(tT), \quad \overline{\mathbf{Q}}^{(T)}(t) = T^{-1/2}\mathbf{Q}^{(T)}(tT)$$

by some limiting processes to the approximation of $\mathbf{X}^{(T)}(\cdot)$, $\mathbf{Y}^{(T)}(\cdot)$, and $\mathbf{Q}^{(T)}(\cdot)$ by the accompanying diffusion processes $\hat{\boldsymbol{\xi}}^{(T)}(\cdot)$, $\hat{\boldsymbol{\eta}}^{(T)}(\cdot)$, and $\hat{\boldsymbol{\zeta}}^{(T)}(\cdot)$, respectively.

Remark 5.4. With Corollary 5.2 established, the proof of the limit theorems of diffusion approximation in [12, 4] may be simplified. It reduces to studying the convergence of the Brownian process $T^{-1/2}\hat{\boldsymbol{\xi}}^{(T)}(tT)$ to some limiting Brownian motion.

The rest of this section is devoted to proofs of the stated results. We precede the proof of Theorem 5.1 by several auxiliary lemmas.

Lemma 5.1. *If the first condition in (5.13) holds for some $j \in J$ then*

$$\sup_{t\le T} |A_{0j}^{(T)} - t\lambda_{0j}(T)| = O\Big((T\lambda_{0j}(T)\sigma_{0j}^2(T))^{1/2}\Big). \tag{5.23}$$

This relation is trivial under $\lambda_{0j}(T) = 0$, since in this case $A_{0j}^{(T)}(\cdot) \equiv 0$. If $\lambda_{0j}(T) > 0$, then the proposition (5.23) follows from Lemma 6.1 with $v^{(T)}(\cdot) \equiv u_j^{(T)}(\cdot)$, $\mu(T) = \lambda_{0j}(T)$, $\sigma(T) = \sigma_{0j}(T)$, $N^{(T)}(\cdot) = A_{0j}^{(T)}(\cdot)$.

Lemma 5.2. *If the second condition in (5.13) holds for some $j \in J$ then*

$$\|\tilde{S}_j^{(T)}\|_T = O\Big((T\mu_j(T)\sigma_j^2(T))^{1/2}\Big) = o(T\mu_j(T)). \tag{5.24}$$

The proposition follows from Lemma 6.1, if we put $v^{(T)}(\cdot) \equiv v_j^{(T)}(\cdot)$, since in this case $\mu(T) = \mu_j(T)$, $\sigma(T) = \sigma_j(T)$ and $N^{(T)}(\cdot) = S_j^{(T)}(\cdot)$.

Lemma 5.3. *If Lemma 5.2 holds, then*

$$\|\tilde{R}_{ij}^{(T)}(\cdot)\|_{S_i^{(T)}(T)} = O\Big((T\mu_j(T))^{1/2}\Big). \tag{5.25}$$

Proof. Not e that $\tilde{R}_{ij}^{(T)}(n)$ is a sum of n i.i.d. random variables with zero expectations and variances $p_{ij}(1-p_{ij})$. Consequently, by the Kolmogorov inequality

$$\mathbf{P}\Big(\|R_{ij}^{(T)}(\cdot)\|_n \ge x\Big) \le \frac{np_{ij}(1-p_{ij})}{x^2}.$$

Hence, for all $n = n(T)$,

$$\|R_{ij}^{(T)}(\cdot)\|_n =_p O(n^{1/2}). \tag{5.26}$$

Now (5.26) implies (5.25), since

$$\mathbf{P}\Big(S_j^{(T)}(T) \ge T\mu_j(T)\Big) \to 0 \tag{5.27}$$

as it follows from (5.24).

To establish Theorem 5.1, it is sufficient to apply Theorem 4.1 and estimate in it $\{\alpha_j(T)\}$ using inequalities (4.8), (5.12), and (5.23)–(5.25).

For the proof of Theorem 5.3 we need several auxiliary lemmas.

Lemma 5.4. *Suppose that conditions (5.15), (5.17), and (5.13) hold for some j. Then we can construct a Wiener process $W_{0j}^{(T)}(\cdot)$, such that (5.18) holds.*

To verify this proposition, Theorem 6.1 can be applied with $v^{(T)}(\cdot) \equiv u_j^{(T)}(\cdot)$.

Lemma 5.5. *Suppose that conditions (5.16), (5.17), and (5.13) hold for some j. Then we can construct a Wiener process $W_j^{(T)}(\cdot)$, such that (5.19) holds.*

This Lemma follows from Theorem 6.1 with $v^{(T)}(\cdot) \equiv v_j^{(T)}(\cdot)$.

Lemma 5.6. *Suppose that conditions (5.17) hold for some j. Then we can construct a Wiener processes $W_{jk}^{(T)}$, $k \in J \cup \{0\}$, such that (5.20) holds.*

Proof. By definitions (3.2) and (4.2), the random vector $R_{j\bullet}^{(T)}(n)$ with the components $R_{jk}^{(T)}(n)$, $k \in J \cup \{0\}$ is a sum of n independent restricted random vectors with the components $I\{r_j^{(T)}(\cdot) = k\}$, $k \in J \cup \{0\}$. From (5.4) it follows that these vectors are i.i.d. and their distributions do not depend on T. Moreover, their expectation and covariance matrix are the same as of the vector with the components

$$p_{jk} + p_{jk}^{1/2} W_{jk}^{(T)}(1) + p_{jk} \overline{W}_j^{(T)}(1), \quad k \in J \cup \{0\}.$$

By the Einmahl theorem [8], which is a multidimensional generalization of KMT approximation, we may construct Wiener processes $W_{jk}^{(T)}(\cdot)$, such that

$$\gamma_{jk}(T) =_p O(\ln(1 + T\mu_j(T))) = O(r(T)).$$

Put

$$\xi_{0j}(t) = t\lambda_{0j}(T) + \lambda_{0j}^{1/2}(T)\sigma_{0j}(T)W_{0j}^{(T)}(t),$$

$$\tilde{\xi}_j(t) = \mu_j^{1/2}(T)\sigma_j(T)W_j^{(T)}(t), \quad \breve{\xi}_{jk}^{(T)}(t) = p_{jk}^{1/2} W_{jk}^{(T)}(t) - p_{jk}\overline{W}_j^{(T)}(t). \tag{5.28}$$

Lemma 5.7. *If conditions (5.12), (5.13), and (5.17) hold then*

$$\beta_j(T) \equiv \beta\Big(\alpha_j(T); T, \tilde{\xi}_j(\cdot)\Big) =_p O(r(T)), \tag{5.29}$$

$$\beta_{jk}(T) \equiv \beta\Big(\mu_i\alpha_i(T); \mu_i(T), \breve{\xi}_{jk}(\cdot)\Big) =_p O(r(T)), \tag{5.30}$$

$$\breve{\beta}_{jk}(T) \equiv \beta\Big(\|\tilde{S}_i(\cdot)\|_T; \mu_i(T), \breve{\xi}_{jk}(\cdot)\Big) =_p O(r(T)). \tag{5.31}$$

Proof. From Theorem 5.1 and Lemma 6.3 we obtain

$$\beta_j(T) = \mu_j^{1/2}(T)\sigma_j(T)\beta(O(r_0(T)/\mu_j(T)), T, W_j^{(T)}(\cdot)) =_p$$

$$=_p O(\sigma_j(T) r_0^{1/2}(T) \ln^{1/2}(1 + T\mu_j(T)/r_0(T))). \quad (5.32)$$

Inequality (5.30) follows similarly, since for any Wiener process $W(\cdot)$,

$$\beta(\mu_j(T)\alpha_j(T); T\mu_j(T), W(\cdot)) = \beta(O(r_0(T)); T\mu_j(T), W(\cdot)) =$$

$$= O(r_0^{1/2}(T) \ln^{1/2}(1 + T\mu_j(T)/r_0(T))). \quad (5.33)$$

Relation (5.31) is obtained analogously, since $\|\tilde{S}_j^{(T)}(\cdot)\|_T =_p O(r_0(T))$ by (5.24) and (5.12).

Applying (5.12) and (5.17) we obtain (5.29) – (5.31) from (5.32) and (5.33).

Proof of Theorem 5.3. We use Theorem 4.3 and Remark 4.9 with $S_j(0) = Q_j(0)$, and $\xi_{0j}(\cdot)$, $\tilde{\xi}_j(\cdot)$, $\breve{\xi}_{ij}(\cdot)$ defined in (5.28). In this case, the estimate

$$\Delta_{0j}(T; \hat{\xi}) \leq \gamma_{0j}(T) + \gamma_j(T) + \beta_j(T) \quad (5.34)$$

follows from definitions (4.12), (5.18), (5.19), and (5.29).

Similarly, the inequality

$$\Delta_{ij}(T; \hat{\xi}) \leq \gamma_{ij}(T) + \breve{\beta}_{ij}(T) + \gamma_i(T) + \beta_{ij}(T) + \beta_i(T) p_{ij}, \quad (5.35)$$

holds by (4.12), (4.23), and (5.18) – (5.20), if $S_i^{(T)}(2T) \geq T\mu_i(T)$.

Thus, Theorem 5.3 follows from estimates (5.34), (5.35), (5.27), and Lemmas 5.4 – 5.7.

From definitions (5.28), (5.9), and (5.18) – (5.22), we have

$$\|\hat{X}_j(\cdot) - \hat{\xi}_j(\cdot)\| \leq \gamma_{0j}(T) + \gamma_j(T) + \sum_i \gamma_i(T) p_{ij} + \sum_i \gamma_{ij}(T).$$

Therefore, Theorem 5.2 follows from Theorem 5.3.

All corollaries in Section 5 are direct consequences of corresponding theorems.

6. Appendix

Using ideas from the works [6, 10, 13] we prove in this section several auxiliary results for approximations of renewal processes which we have exploited in Section 5. For each $T > 0$ consider an infinite sequence $v^{(T)}(1), v^{(T)}(2), \ldots$ of i. i. d. random variables satisfying the following conditions

$$0 < \mathbf{E} v^{(T)}(1) = 1/\mu(T) < \infty, \quad \sigma^2(T) = \mathbf{E}(\tilde{v}^{(T)}(1))^2 < \infty,$$

where

$$\tilde{v}^{(T)}(i) = \mu(T) v^{(T)}(i) - 1, \quad i = 1, 2, \ldots .$$

For $t \geq 0$, put

$$V^{(T)}(t) = \sum_{i \leq t} v^{(T)}(i), \quad N^{(T)}(t) = \min\{k : V^{(T)}(k) \geq t\}. \tag{6.1}$$

First, let us prove one rough estimate. Put

$$\tilde{V}^{(T)}(x) = \mu(T)V^{(T)}(x) - x, \quad \tilde{N}^{(T)}(t) = t\mu(T) - N^{(T)}(t).$$

Lemma 6.1. *Suppose that*

$$\sigma^2(T) = o(T\mu(T)). \tag{6.2}$$

In this case

$$\|\tilde{N}^{(T)}\|_T =_p O\Big((T\mu(T)\sigma^2(T))^{1/2}\Big). \tag{6.3}$$

Proof. Following the proof of Lemma from [10], it is easy to show that $\|\tilde{N}^{(T)}(\cdot)\|_T \leq \|\tilde{V}^{(T)}(\cdot)\|_{2T\mu(T)}$, if the inequality $N^{(T)}(T) \leq 2T\mu(T)$ is valid. Hence

$$\mathbf{P}\Big(\|\tilde{N}^{(T)}(\cdot)\|_T \geq x(T\mu(T)\sigma^2(T))^{1/2}\Big) \leq \mathbf{P}\Big(N^{(T)}(T) \geq 2T\mu(T)\Big)+$$

$$+\mathbf{P}\Big(\|\tilde{V}^{(T)}(\cdot)\|_{2T\mu(T)} \geq x(T\mu(T)\sigma^2(T))^{1/2}\Big) \tag{6.4}$$

Note that $\tilde{V}^{(T)}(n)$ is a sum of n independent random variables $\{\tilde{v}^{(T)}(\cdot)\}$ with zero means. By the Kolmogorov inequality, we have

$$\mathbf{P}\Big(\|\tilde{V}^{(T)}(\cdot)\|_{2T\mu(T)} \geq x\Big) \leq 2T\mu(T)\sigma^2(T)/x^2. \tag{6.5}$$

In particular, for $x = T\mu(T)$ we obtain from (6.1) and (6.5)

$$\mathbf{P}\Big(N^{(T)}(T) \geq 2T\mu(T)\Big) \leq \mathbf{P}\Big(V^{(T)}(2T\mu(T)) \leq T\Big) =$$

$$= \mathbf{P}\Big(\tilde{V}^{(T)}(2T\mu(T)) \geq T\mu(T)\Big) \leq \sigma^2(T)/(T\mu(T)). \tag{6.6}$$

From (6.4) – (6.6) we have

$$\mathbf{P}\Big(\|\tilde{N}^{(T)}(\cdot)\|_T \geq x(T\mu(T)\sigma^2(T))^{1/2}\Big) \leq 2\sigma^2(T)/(T\mu(T)) + 2/x^2. \tag{6.7}$$

Relation (6.3) follows from (6.7).

Theorem 6.1. *Suppose that condition (6.2) is satisfied. Let* $r(T) > 0$ *and* $s \geq 2$ *be such that*

$$T\mu(T)\mathbf{E}\min\left\{\frac{|\tilde{v}^{(T)}(1)|^s}{r^s(T)},\ \frac{|\tilde{v}^{(T)}(1)|^2}{r^2(T)}\right\} \to 0 \tag{6.8}$$

and

$$\sigma^2(T)\ln(1+T\mu(T)/r(T)) = o(r(T)). \tag{6.9}$$

Then on the same probability space where the random process $N^{(T)}(\cdot)$ is defined, we can construct a Wiener process $W^{(T)}(\cdot)$, such that

$$\delta_0(T) \equiv \sup_{0\le t\le T} |N^{(T)}(t) - t\mu(T) - \mu^{1/2}(T)\sigma(T)W^{(T)}(t)| =_p O(r(T)). \tag{6.10}$$

Remark 6.1. If $\sigma(T) = 0$ then $N^{(T)}(t) \equiv T\mu(T)$, and the right-hand side of (6.10) vanishes.

Before proving Theorem 6.1 we establish several auxiliary propositions.

Lemma 6.2. *Assume that (6.8) holds. Then we can construct a Wiener process $W_1^{(T)}(\cdot)$, such that the difference*

$$\varepsilon_1^{(T)}(t) = \mu(T)V^{(T)}(t) - t - \sigma(T)W_1^{(T)}(t) \tag{6.11}$$

satisfies the relation

$$\|\varepsilon_1^{(T)}(\cdot)\|_{2\mu(T)T} =_p O(r(T)). \tag{6.12}$$

Proof. Note that $V^{(T)}(t) - t/\mu(T)$ is a random step function, constructed by sums of i. i. d. random variables $\{\tilde{v}^{(T)}(\cdot)\}$ with zero expectations and variances $\sigma^2(T)$. As shown in [13], for all $T_1 > 0$, $x > 0$ and $s \ge 2$ we can choose a Wiener process $W_1^{(T)}(\cdot)$ so that

$$\mathbf{P}(\|\varepsilon_1^{(T)}(cdot)\|_{T_1} \ge x) \le C_1(s)T_1\mathbf{E}\min\Big\{|\tilde{v}^{(T)}(1)|^s/x^s, \quad |\tilde{v}^{(T)}(1)|^2/x^2\Big\}. \tag{6.13}$$

Relation (6.12) follows from (6.13) with $T_1 = 2T\mu(T)$ and $x = r(T)$.

As in (1.3), let $\beta(\delta; U, W(\cdot))$ be the maximal oscillation of a Wiener process on the interval $[0, U]$. From Lemma 1.2.1 in [7] we obtain

Lemma 6.3. *For all numbers $\delta = \delta(T) > 0$ and $U = U(T) > 0$*

$$\beta(\delta; U, W(\cdot)) =_p O\Big((\delta\ln(1+U/\delta))^{1/2}\Big), \quad \|W(\cdot)\|_U =_p O\Big(U^{1/2}\Big).$$

Now assume that

$$\sigma(T) > 0 \tag{6.14}$$

and denote

$$V_0^{(T)}(x) = \mu(T)V^{(T)}(x\sigma^2(T))/\sigma^2(T), \quad N_0^{(T)}(t) = \min\Big\{x : V_0^{(T)}(x) \ge t\Big\}. \tag{6.15}$$

Lemma 6.4. *Let conditions (6.2), (6.8), and (6.14) be satisfied. Then we can define a Wiener process* $\hat{W}_0^{(T)}(\cdot)$*, so that the difference*

$$\hat{\varepsilon}_0^{(T)}(t) = N_0^{(T)}(t) - t - \hat{W}_0^{(T)}(t), \tag{6.16}$$

satisfies the relation

$$\|\hat{\varepsilon}_0^{(T)}(\cdot)\|_{U(T)} =_p O(r(T)/\sigma^2(T) + \ln(1 + U(T)) + \ln(1 + T\mu(T)/r(T)), \tag{6.17}$$

where $U(T) = T\mu(T)/\sigma^2(T)$.

Proof. From (6.11), (6.15,) and (6.16) the next representation follows

$$V_0^{(T)}(t) = t + W_0^{(T)}(t) + \varepsilon_0^{(T)}(t) \quad \text{for} \quad 0 \leq t \leq 2U(T),$$

where

$$W_0^{(T)}(t) = W_1^{(T)}(t\sigma^2(T))/\sigma(T) \quad \text{and} \quad \varepsilon_0(t) = \varepsilon_1(t\sigma^2(T))/\sigma^2(T). \tag{6.18}$$

Since the process $W_0^{(T)}(\cdot)$ defined in (6.18) is a standard Wiener process, we may follow the proof of Theorem 4.1 from [6] with $\mu = \sigma = 1$. As a result, on the same probability space where the process $W_0^{(T)}(\cdot)$ is defined, we can construct another standard Wiener process $\hat{W}_0^{(T)}(\cdot)$ so that

$$\Big(\|\hat{\varepsilon}_0^{(T)}(\cdot)\|_{U(T)} =_p O\Big(\|\varepsilon_0^{(T)}(\cdot)\|_{2U(T)} + \ln(1 + U(T)) + \beta_0(T)\Big)\Big). \tag{6.19}$$

Here $\beta_0(T) = \beta\Big(\|\varepsilon_0^{(T)}(\cdot)\|_{2U(T)};\, 2U(T),\, W_0(\cdot)\Big)$. By Lemma 6.2 and definition (6.15), we have

$$\|\varepsilon_0^{(T)}(\cdot)\|_{2U(T)} =_p O(r(T)/\sigma^2(T)) \tag{6.20}$$

and from Lemma 6.3 we obtain

$$\begin{aligned}\beta_0(T) &=_p O\Big(\Big(r(T)\sigma^{-2}(T)\ln(1 + U(T)\sigma^2(T)/r(T)\Big)\Big)^{1/2} \\ &=_p O\Big(r(T)/\sigma^2(T) + \ln(1 + T\mu(T)/r(T))\Big).\end{aligned} \tag{6.21}$$

Now (6.19) – (6.21) imply (6.17).

Finally, we prove Theorem 6.1. First of all, note that

$$N^{(T)}(t) = \sigma^2(T)N_0^{(T)}\Big(t\mu(T)/\sigma^2(T)\Big) \quad \text{and} \quad \delta_0(T) = \sigma^2(T)\|\hat{\varepsilon}_0^{(T)}(\cdot)\|_{U(T)} \tag{6.22}$$

from definitions (6.10) and (6.15). Now the assertion of Theorem 6.1 follows from (6.22), (6.17) and (6.9), when auxiliary condition (6.14) is satisfied. The case when condition (6.14) fails was considered in Remark 6.1.

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