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# ON CONDITIONS FOR CONVERGENCE OF THE DENSITIES OF SMOOTHED DISTRIBUTIONS IN THE CENTRAL LIMIT THEOREM

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## 1. Introduction

Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of independent identically distributed random variables such that

$$\mathbf{M}\xi = 0, \quad \mathbf{D}\xi = 1. \tag{1}$$

Put $S_n = \xi_1 + \dots + \xi_n$, and denote by $\eta$ a random variable with standard normal distribution.

The technique of smoothing is employed somehow in many articles devoted to studying the rate of convergence of the distribution of $S_n/\sqrt{n}$ to the normal law. We dwell on one of such results.

Suppose that a random variable $\nu$ is independent of $S_n$ and $\eta$ and has a density $v(x)$. Then the random variable $(S_n + \nu)/\sqrt{n}$ has the density

$$f_n(x) = \sqrt{n}\mathbf{M}v(\sqrt{n}x - S_n). \tag{2}$$

Since by the Central Limit Theorem the distribution of $S_n/\sqrt{n}$ is close to that of $\eta$, the following natural question arises: under which conditions the density $f_n(x)$ is close to the normal density $\varphi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ of the distribution of $\eta$ or to the density $\varphi_n(x)$ of the distribution of $\eta + \nu/\sqrt{n}$ which obviously equals

$$\varphi_n(x) = \sqrt{n}\mathbf{M}v(\sqrt{n}x - \eta) = \int_{-\infty}^{\infty} \varphi(x - y/\sqrt{n})v(y)\,dy. \tag{3}$$

The following useful assertion was obtained by V. V. Yurinsky in the article [1] devoted to estimations of the Prokhorov distance between distributions in the Central Limit Theorem. It provides one of the possible answers.

**Theorem A.** *Suppose that the distribution of $\xi$ satisfies (1) and*

$$\mathbf{M}|\xi|^3 = \gamma < \infty, \tag{4}$$

*where $\gamma$ is a fixed number. Then there exist a density $v(\cdot)$ and a number $C > 0$ such that*

$$\int_{-\infty}^{\infty} |f_n(x) - \varphi_n(x)|\,dx \le C\gamma/\sqrt{n}. \tag{5}$$

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In the proof, V. V. Yurinsky used a specific density $v(\cdot)$ of the form $v(x) = v_0(x/\gamma)/\gamma$ which, in particular, satisfies the condition

$$\hat{v}(t) = 0 \quad \text{for} \quad |t| \geq c_0/\gamma, \tag{6}$$

where $c_0 > 0$ and

$$\hat{v}(t) = \mathbf{M}e^{it\nu} = \int_{-\infty}^{\infty} e^{itx} v(x)\, dx \tag{7}$$

is the characteristic function of the random variable $\nu$.

Since condition (6) on the smoothing density is often of purely technical nature, it seems to be a natural step to eliminate it in order to enlarge the class of possible smoothing densities. However, the following assertion shows that (6) is in a certain sense necessary for (5) to be valid.

**Theorem 1.** *Suppose that the density $v(\cdot)$ and the number $\gamma$ are such that (5) holds for all $n$ and for all possible distributions of a random variable $\xi$ satisfying (1) and (4). If*

$$\gamma \geq 7^{3/2}/3^{5/2} \approx 1,19\ldots, \tag{8}$$

*then (6) is necessarily true with $c_0 = 2\pi$.*

This result, surprising in the authors' opinion, will emerge after we examine in § 2 the behavior of $f_n(x)$ in the case of an arbitrary lattice distribution of $\xi$ along the lines of [2].

## 2. Asymptotic Properties of Densities of Smoothed Lattice Distributions

Suppose that a random variable $\xi$ satisfies (1) and has a lattice distribution with maximal span $h$. In this case for some $a \in [0, h)$ and for all $n$

$$\sum_m \mathbf{P}(S_n = na + mh) = 1.$$

Also, by the local limit theorem (see [3, p. 231] and [4, p. 152])

$$\begin{aligned} \alpha_n &\equiv \sup_m |\sqrt{n}\mathbf{P}(S_n = na + mh) - h\varphi(x_{n,m})| \to 0, \\ \beta_n &\equiv \sum_m |\mathbf{P}(S_n = na + mh) - n^{-1/2} h\varphi(x_{n,m})| \to 0 \end{aligned} \tag{9}$$

as $n \to \infty$, where $x_{n,m} = (na + mh)/\sqrt{n}$, with $\sum\limits_m$ standing for summation over all integers $m$ here and in the sequel.

For $t \in [0, h)$ put

$$v_h(t) = \sum_m v(mh + t), \quad v_{n,h}(t) = \sum_m \min\{1, mh/\sqrt{n}\} v(mh + t). \tag{10}$$

Since

$$\int_0^h v_h(t)\, dt = \sum_m \int_m^{m+h} v(x)\, dx = \int_{-\infty}^{\infty} v(x)\, dx = 1, \tag{11}$$

the function $v_h(\cdot)$ is the density of a probability distribution concentrated on the interval $[0, h)$. In line with [5, p. 270] we assume the density $v_h(\cdot)$ to be obtained by "winding" the density $v(\cdot)$ on a circle of length $h$. In particular, from (11) we see that for almost every $t \in [0, h)$

$$v_h(t) < \infty. \tag{12}$$

For every $t \in [0, h)$ such that (12) holds, put

$$\begin{aligned} \Delta_{n,m}(t) &= f_n(x_{n,m} + t/\sqrt{n}) - h\varphi(x_{n,m})v_h(t), \\ \Delta_n(t) &= n^{-1/2} \sum_m |\Delta_{n,m}(t)|. \end{aligned} \tag{13}$$

**Theorem 2.** *If $t \in [0, h)$ satisfies (12) then*

$$\sup_m |\Delta_{n,m}(t)| \le \alpha_n v_h(t) + \varphi(0) h v_{n,h}(t), \tag{14}$$

$$\Delta_n(t) \le (\beta_n + 2n^{-1/2}) v_h(t) + 2v_{n,h}(t). \tag{15}$$

Observe that the inequalities (14) and (15) remain consistent for all $t \in [0, h)$, if only we assume that

$$|\Delta_{n,m}(t)| = \Delta_n(t) = \infty \quad \text{if} \quad v_h(t) = \infty.$$

Below we present a series of simple corollaries to Theorem 2.

**Corollary 1.** *For almost every $t \in [0, h)$*

$$\sup_m |\Delta_{n,m}(t)| \to 0 \quad \textit{and} \quad \Delta_n(t) \to 0 \quad \textit{as} \quad n \to \infty.$$

**Corollary 2.** *As $n \to \infty$*

$$\int_{-\infty}^{\infty} |f_n(x) - \varphi(x)|\, dx \to \int_0^h |v_h(t) - h^{-1}|\, dt, \tag{16}$$

$$\int_{-\infty}^{\infty} |f_n(x) - \varphi_n(x)|\, dx \to \int_0^h |v_h(t) - h^{-1}|\, dt. \tag{17}$$

**Corollary 3.** *If*

$$\int_{-\infty}^{\infty} |f_n(x) - \varphi_n(x)|\, dx \to 0 \quad \textit{as} \quad n \to \infty, \tag{18}$$

*then*

$$v_h(t) = h^{-1} \quad \textit{for almost every} \quad t \in [0, h) \tag{19}$$

*and, in particular,*

$$\hat{v}(2\pi k/h) = 0 \quad \textit{for} \quad k = \pm 1, \pm 2, \pm 3, \ldots. \tag{20}$$

Thus, convergence in variation (18) of the smoothed distributions implies the vanishing of the characteristic function of the smoothing density at some points.

Observe that whether (18) holds or fails depends solely on the distribution of $\xi$ and on the density $v(\cdot)$. Moreover, in view of Corollary 2 the fact that (18) holds for some lattice distribution of $\xi$ having maximal span $h$ and satisfying (1) implies that (18) is true for all such distributions with the same span $h$, if only the density $v(\cdot)$ is fixed.

We now strengthen Corollary 3 in the case when a sufficiently large class of distributions of $\xi$ meets (18).

**Corollary 4.** *Suppose that a density $v(\cdot)$ and a number $H > 0$ are such that for all $h \in (H/2, H)$ there exists a random variable $\xi$ satisfying (1), having a lattice distribution with maximal span $h$, and such that (18) holds. Then*

$$\hat{v}(t) = 0 \quad \textit{if} \quad |t| \geq 2\pi/H. \tag{21}$$

Theorem 1 of § 1 will be obtained as a particular case of Corollary 4 for $H = \gamma$.

Condition (8) in Theorem 1 is essential, as illustrated by the following example.

**Example**. Let the random variable $\xi$ take the values $+1$ and $-1$ with equal probability and let $v(\cdot)$ be the density of the uniform distribution on the interval $[b, b+2l]$, where $l$ is a natural number. It is easy to verify that in this case (18)–(20) hold for $h = 2$. Moreover, using the estimates of [3, p. 255] instead of (9) in the local limit theorem, we easily see that

$$\int_{-\infty}^{\infty} |f_n(x) - \varphi_n(x)|\, dx \leq C(l)/\sqrt{n};$$

i.e. (5) holds also for all $n$.

**Remark**. For $\gamma = 1$ Theorem 1 fails. Indeed, in this case it only the distribution of $\xi$ presented in the above example that satisfies both (1) and (4). Consequently, for $\gamma = 1$ inequality (5) is true, for instance, for the density $v(\cdot)$ of the uniform distribution on the interval $[-1, 1]$ for which $\hat{v}(t) = t^{-1} \sin t$.

We have thus shown that Theorem 1 is not true without condition (8). However, the question remains open whether the constant on the right-hand side of (8) can be diminished.

## 3. Proofs

For notational simplicity, we put

$$\begin{aligned} p_k &= \mathbf{P}(S_n = na + kh) = \mathbf{P}(S_n/\sqrt{n} = x_{n,k}), \\ \varphi_k &= n^{-1/2} h \varphi(x_{n,k}), \quad \varepsilon_k = \sum_m |\varphi_m - \varphi_{m-k}|. \end{aligned} \tag{22}$$

Using this, from (9) and (2) we have

$$\begin{aligned} \alpha_n &= \sqrt{n} \underset{l}{\rightarrow} \sup |p_l - \varphi_l|, \quad \beta_n = \sum_l |p_l - \varphi_l|, \\ f_n(x_{n,m} + t/\sqrt{n}) &= \sqrt{n} \sum_l p_l v(mh - lh + t). \end{aligned} \tag{23}$$

Consequently, by definitions (9) and (13)

$$n^{-1/2} \Delta_{n,m}(t) = \sum_k p_{m-k} v(kh + t) - \varphi_m \sum_k v(kh + t). \tag{24}$$

We first prove several auxiliary statements. Put

$$\delta_{n,m}(t) = \varphi(x_{n,m} + t/\sqrt{n}) - \varphi(x_{n,m}), \quad \delta_n(t) = n^{-1/2} \sum_m |\delta_{n,m}(t)|. \tag{25}$$

**Lemma.** *The following estimates take place:*

$$\underset{t}{\in} [0,h) \to \sup \delta_n(t) \le 2\varphi(0)/\sqrt{n}, \tag{26}$$

$$\Big|\sum_m \varphi_m - 1\Big| \le 2\varphi(0)/\sqrt{n}, \tag{27}$$

$$\varepsilon_k \le 4\varphi(0)/\sqrt{n} + \min\{2, 2\varphi(0)kh/\sqrt{n}\}, \tag{28}$$

$$\underset{m}{\to} \sup |\varphi_m - \varphi_{m-k}| \le \varphi(0)(h/\sqrt{n}) \min\{1, e^{-1/2}kh/\sqrt{n}\}, \tag{29}$$

$$\int\limits_{-\infty}^{\infty} |\varphi_n(x) - \varphi(x)|\,dx \le 2\mathbf{M} \min\{1, \varphi(0)\nu/\sqrt{n}\}. \tag{30}$$

*Proof.* From the explicit form of the functions $\varphi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and $\varphi'(x) = -x\varphi(x)$ it is easy to note that

$$C_1 \equiv \underset{x}{\to} \sup \varphi(x) = \varphi(0) = (2\pi)^{-1/2}, \quad C_2 \equiv \underset{x}{\to} \sup |\varphi'(x)| = |\varphi'(1)| = e^{-1/2}\varphi(0),$$

$$C_3 \equiv \int\limits_{-\infty}^{\infty} |\varphi'(x)|\,dx = 2\varphi(0) < 1. \tag{31}$$

Furthermore, if $t \in [0,h)$ then by (25)

$$\sqrt{n}\delta_n(t) = \sum_m \Bigg| \int\limits_0^{t/\sqrt{n}} \varphi'(x_{n,m} + x)\,dx \Bigg| \le \int\limits_{-\infty}^{\infty} |\varphi'(x)|\,dx = C_3. \tag{32}$$

From (31) and (32) we infer (26). Moreover,

$$\Big|\sum_m \varphi_m - 1\Big| = \Bigg| \int\limits_{-\infty}^{\infty} \varphi(x)\,dx - \sum_m \varphi_m \Bigg| = \Bigg| \sum_m \int\limits_0^h \delta_{n,m}(t)\,dt/\sqrt{n} \Bigg| \le \delta_n(h),$$

implying (27).

Comparing the definitions (22) and (25), we can see that

$$\varepsilon_k \le k\varepsilon_1 = kh\delta_n(h) \le C_3kh/\sqrt{n}. \tag{33}$$

On the other hand, by (22) and (27)

$$\varepsilon_k \le \sum_m \varphi_m + \sum_m \varphi_{m-k} \le 2 + 2C_3/\sqrt{n}. \tag{34}$$

From (33) and (34) we immediately derive (28).

For an arbitrary $y$,

$$\varepsilon(y) \equiv \int\limits_{-\infty}^{\infty} |\varphi(x-y) - \varphi(x)|\,dx \le \int\limits_0^1 \left( \int\limits_{-\infty}^{\infty} |y||\varphi'(x-ty)|\,dx \right) dt = C_3|y|. \tag{35}$$

On the other hand,

$$\varepsilon(y) \le \int_{-\infty}^{\infty} \varphi(x-y)\,dx + \int_{-\infty}^{\infty} \varphi(x)\,dx = 2. \tag{36}$$

From (35), (36), and (31) we have

$$\varepsilon(y) \le \min\{2,\, C_3|y|\} = 2\min\{1,\, \varphi(0)|y|\}. \tag{37}$$

Observe that by (3) the left-hand side of (30) does not exceed $\mathbf{M}\varepsilon(\nu/\sqrt{n})$. From this fact and (37) we infer (30).

To prove (29), we apply (22), (31), and the elementary inequality $|\varphi(x)-\varphi(x-y)| \le \min\{C_1,\, C_2|y|\}$, with $x = x_{n,m}$ and $y = kh/\sqrt{n}$.

*Proof of Theorem 2.* By (24)

$$|\Delta_{n,m}(t)| \le \sqrt{n}\sum_k |p_{m-k} - \varphi_{m-k}|v(kh+t) + \sqrt{n}\sum_k |\varphi_m - \varphi_{m-k}|v(kh+t). \tag{38}$$

The inequality (14) follows immediately from (38), (27) and the definition (23) of $\alpha_n$.

Summing up (38) over all $m$ and recalling the definitions of $\Delta_n(t)$, $\beta_n$ and $\varepsilon_k$, we obtain

$$\Delta_n(t) \le \sum_k \beta_n v(kh+t) + \sum_k \varepsilon_k v(kh+t). \tag{39}$$

From (39) and (28) we infer (15).

*Proof of Corollary 1.* By (10) all terms of the series $v_{n,h}(t)$ vanish as $n \to \infty$, and the series is dominated by the series $v_h(t)$. Consequently,

$$v_h(t) \ge v_{n,h}(t) \to 0 \quad \text{as} \quad n \to \infty. \tag{40}$$

The claim of Corollary 1 is a consequence of (40), (14) and (15).

Note that by (40) and (11) the Lebesgue Dominated Convergence Theorem yields the convergence

$$\int_0^h v_{n,h}(t)\,dt \to 0 \quad \text{as} \quad n \to \infty. \tag{41}$$

*Proof if Corollary 2.* Introduce the notation

$$\begin{gathered} \rho_{n,m}(t) = f_n(x_{n,m} + t/\sqrt{n}) - \varphi(x_{n,m} + t/\sqrt{n}), \\ \rho_n(t) = n^{-1/2}\sum_m |\rho_{n,m}(t)|, \quad \rho = \int_0^h |h^{-1} - v_h(t)|\,dt. \end{gathered} \tag{42}$$

Representing the real axis as a union of intervals of the form $[x_{n,m}, x_{n,m+1})$, we see that

$$\rho_n \equiv \int_{-\infty}^{\infty} |f_n(x) - \varphi(x)|\,dx = \int_0^h \rho_n(t)\,dt. \tag{43}$$

Definitions (13), (25), and (42) yield the identity

$$\Delta_{n,m}(t) = \rho_{n,m}(t) + \delta_{n,m}(t) + \varphi(x_{n,m})(1 - hv_h(t)). \tag{44}$$

With (44) taken into account, the following inequality is straightforward:

$$|\,|\rho_{n,m}(t)| - h\varphi(x_{n,m})|h^{-1} - v_h(t)|\,| \le |\Delta_{n,m}(t)| + |\delta_{n,m}(t)|. \tag{45}$$

Recalling the notations of (22) and (42), from (43) and (45) we have

$$|\rho_n - \sum_m \varphi_m \rho| \le \int_0^h \Delta_n(t)\,dt + \int_0^h \delta_n(t). \tag{46}$$

To derive (16) from (46), it suffices to observe that by (26) and (41) the right-hand side of (46) vanishes as $n \to \infty$ and that by (27) $\sum_m \varphi_m \to 1$.

Convergence (17) ensues from (16) and (30).

*Proof of Corollary 3.* If (18) is satisfied then (19) is apparent from (17).

Furthermore, by definitions (7) and (10) we have

$$\begin{aligned} \hat{v}(2\pi kh) &= \int_{-\infty}^{\infty} e^{i2\pi kx/h} v(x)\,dx \\ &= \sum_m \int_0^h e^{i2\pi km + i2\pi kt/h} v(mh+t)\,dt = \int_0^h e^{i2\pi kt/h} v_h(t)\,dt. \end{aligned} \tag{47}$$

With (19) holding, from (47) we obtain

$$\hat{v}(2\pi k/h) = \int_0^h e^{i2\pi kt/h}\,dt/h = \int_0^{2\pi} (\cos kx + i \sin kx)\,dx = 0$$

for $k = \pm 1, \pm 2, \dots$, which proves (20).

*Proof of Corollary 4.* Under the assumptions made and in view of (20)

$$\hat{v}(t) = 0 \quad \text{for all} \quad t \in U, \quad \text{where} \quad U = \bigcup_{k \ne 0} (2\pi k/H, 4\pi k/H). \tag{48}$$

Since the characteristic function $\hat{v}(t)$ is continuous, the set $U$ in (48) can be replaced by its closure $U_0$. To obtain (21), it remains to observe that $U_0 = (-\infty, -2\pi/H] \cup [2\pi/H, \infty)$.

*Proof of Theorem 1.* We apply Corollary 4 with $H = \gamma$. To prove Theorem 1, it suffices to proceed as follows. Given an $h$ in the interval $(\gamma/2, \gamma)$, construct a symmetrically distributed random variable $\xi$ satisfying (1) and (4), taking the values 0, $\pm h$, and $\pm 2h$, and having maximal span $h$.

Put

$$\mathbf{P}(|\xi| = h) = p_1(h) \equiv h^{-3}(2h - \gamma), \quad \mathbf{P}(|\xi| = 2h) = p_2(h) \equiv h^{-3}(\gamma - h)/4. \tag{49}$$

Verify that $\xi$ is a required random variable. We have to check that the functions $p_1(h)$ and $p_2(h)$ in (49) meet the following conditions for all $h \in (\gamma/2, \gamma)$:

$$\begin{gathered} h^2 p_1(h) + (2h)^2 p_2(h) = 1, \quad h^3 p_1(h) + (2h)^3 p_2(h) = \gamma, \\ p_1(h) > 0, \quad p_2(h) > 0, \quad p_1(h) + p_2(h) \le 1. \end{gathered} \tag{50}$$

It is clear that for such $h$ only the last relation in (50) may be not automatically true. However, since the function $p_1(h) + p_2(h)$ on the interval $(\gamma/2, \gamma)$ attains its maximal value at $h = 9\gamma/14$ while $p_1(9\gamma/14) + p_2(9\gamma/14) = 3^{-5}7^3/\gamma^2$, all relations of (50) hold for $\gamma^2 \ge 3^{-5}7^3 = 343/243$. The so-obtained restriction on $\gamma$ coincides with (8).

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