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ASYMPTOTICALLY NORMAL ESTIMATION OF A PARAMETER IN A LINEAR-FRACTIONAL REGRESSION PROBLEM

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1. Introduction

Suppose that in some experiment we observe a sequence of independent random variables $X_1, X_2, \dots, X_N$ such that the following representation is valid for every i:

$$X_i = \frac{a_i}{1 + b_i\theta} + \sigma_i\xi_i, \tag{1.1}$$

with $\xi_1, \dots, \xi_N$ a sequence of independent identically distributed random variables satisfying the conditions

$$\mathbf{E}\xi_i = 0, \quad \mathbf{D}\xi_i = 1. \tag{1.2}$$

Moreover, the values $a_i > 0$ and $b_i > 0$ are assumed known, while the values of the parameter θ and the variances $\mathbf{D}X_i \equiv \sigma_i^2$ are unknown. The values of the random variables $\xi_1, \dots, \xi_N$ are assumed unknown either.

In this article, we study the problem of estimating the unknown parameter $\theta > 0$ from the observations $X_1, \dots, X_N$. This problem is a particular instance of the nonlinear regression problem which is usually solved by the method of least squares or its modifications. Searching an estimator approximately, we often use linearization methods, the steepest descent method, etc. (see, for instance, [1]) whose implementation requires application of computers in view of a huge number of iterations.

However, it turns out that for a linear-fractional regression problem of the form (1.1) the simple estimator

$$\theta^* = \frac{\sum c_i(a_i - X_i)}{\sum c_i b_i X_i} \tag{1.3}$$

is asymptotically normal under rather general assumptions on the constants $\{c_i\}$. Moreover, in the case when some information on the behavior of the variances $\{\sigma_i\}$ is available we can choose functions $\{\gamma_i(\theta)\}$ so that the "improved" estimator

$$\theta^{**} = \frac{\sum \gamma_i(\theta^*)(a_i - X_i)}{\sum \gamma_i(\theta^*) b_i X_i} \tag{1.4}$$

becomes asymptotically efficient in some sense.

The present article is devoted to constructing a class of these two-step estimators and studying their properties. The main results reside in § 2 in which, for clarity of exposition, we do not pursue the goal of giving most general statements. In

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a more abstract situation, the corresponding assertions are given in § 3 and proven in § 4–§ 6.

We note that the methods of the article extend to the case of a multidimensional parameter when the following representation is valid:

$$X_i = \frac{a_i^0 + \sum\limits_{j=1}^{k} a_i^j \theta_j}{1 + \sum\limits_{j=1}^{k} b_i^j \theta_j} + \sigma_i \xi_i. \tag{1.5}$$

In particular, we can construct estimators for unknown parameters in the Michaelis–Menten equation which is widely used in biochemistry (see, for instance, [2]). These results will be given in the forthcoming articles, wherein (1.1) will appear as a particular case of (1.5) enabling us to illustrate all ideas of our method without obscuring them by bulky matrix notations.

We use the symbol $\sum$ without indices only when summation is carried out over i from 1 to N. Below, unless the contrary is specified, we consider all limits as $N \to \infty$ and use the notation $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/2} dy$ for the distribution function of the standard normal law.

2. The Main Results

In this section, we study the properties of the estimators (1.3) and (1.4) in the case when the following easy conditions are met:

$$\inf_i \min\{a_i, b_i, c_i, \sigma_i\} > 0, \quad \sup_i \max\{a_i, b_i, c_i, \sigma_i\} < \infty. \tag{2.1}$$

Put

$$d^2(\{c_i\}) = d^2_{N,\theta}(\{c_i\}, \{\sigma_i\}) = \frac{\sum c_i^2 (1 + b_i\theta)^2 \sigma_i^2}{\left(\sum c_i a_i b_i (1 + b_i\theta)^{-1}\right)^2}. \tag{2.2}$$

Theorem 1. *If (2.1) is satisfied then*

$$\sup_x \left| \mathbf{P}\left(\frac{\theta^* - \theta}{d(\{c_i\})} < x \right) - \Phi(x) \right| \to 0.$$

Now, consider the behavior of a more complicated estimator θ^{**} given by (1.4). Throughout the article, we suppose that all functions $\{\gamma_i(\theta)\}$ are differentiable with respect to θ and the derivatives $\gamma_i'(\theta)$ satisfy the condition

$$\sup_{\theta/2 \le t \le 2\theta} |\gamma_i'(t)| \le K_i(\theta) < \infty. \tag{2.3}$$

Theorem 2. *Suppose that (2.1) is satisfied and the functions $\{\gamma_i(\theta)\}$ are such that*

$$\inf_i \gamma_i(\theta) > 0, \quad \sup_i (\gamma_i(\theta) + K_i(\theta)) < \infty.$$

Then

$$\sup_x \left| \mathbf{P}\left(\frac{\theta^{**} - \theta}{d(\{\gamma_i(\theta)\})} < x \right) - \Phi(x) \right| \to 0,$$

where $d(\{\gamma_i(\theta)\})$ is determined by (2.2) with $c_i = \gamma_i(\theta)$.

Remark 1. It is clear that the accuracy of the estimators θ^* and θ^{**} is determined by the coefficients $d^2(\{c_i\})$ and $d^2(\{\gamma_i(\theta)\})$. Therefore, the problem appears naturally of minimizing these coefficients. The following equalities are easy to verify for every $C > 0$:

$$d^2_{\text{opt}} \equiv \inf_{\{c_i\}} d^2(\{c_i\}) = \inf_{\{\gamma_i(\theta)\}} d^2(\{\gamma_i(\theta)\}) = d^2(\{\gamma_{\text{opt},i}(\theta, \sigma_i)\}),$$

with

$$\gamma_{\text{opt},i}(\theta, \sigma_i) = C \frac{a_i b_i}{(1 + b_i\theta)^3 \sigma_i^2}. \tag{2.4}$$

We emphasize that C in (2.7) may be an arbitrary positive parameter independent of i.

Remark 2. Suppose that the independent random variables ξ_i have the standard normal distribution. Then the variables X_i are normally distributed with mean $U_i(\theta) = a_i/(1 + b_i\theta)$ and variance σ_i^2. Suppose that the variances σ_i^2 are independent of θ. In this case

$$I_N(\theta) = \sum \frac{(U_i'(\theta))^2}{\sigma_i^2} = \sum \frac{a_i^2 b_i^2}{\sigma_i^2 (1 + b_i\theta)^4}, \tag{2.5}$$

where $I_N(\theta)$ is the Fisher information for the sample $X_1, \ldots, X_N$. Inserting the optimal value $\gamma_i(\theta) = \gamma_{\text{opt},i}(\theta, \sigma_i)$ of Remark 1 in the coefficient of the asymptotic variance and using (2.5), we obtain

$$d^2_{\text{opt}} = 1/I_N(\theta). \tag{2.6}$$

Relation (2.6) shows that, by analogy with the Cramér–Ráo inequality, we should expect the estimator θ^{**} to be in a sense unimprovable when $\gamma_i(\theta)$ are chosen optimally.

Example 1. Suppose that

$$\sigma_i^2 = w_{oi}(1 + b_i\theta)^{-3}\sigma^2,$$

where the coefficient $w_{oi} > 0$ is assumed to be known, while the parameter $\sigma > 0$ may be unknown. Then, by Remark 1 on Theorem 1, we can choose optimal constants c_i by putting $c_i = a_i b_i / w_{oi}$.

Remark 3. Using Remark 1, we can easily verify that Example 1 gives the only case in which the optimal values c_i are constants rather than functions of θ and σ_i.

Example 2. Suppose that

$$\sigma_i^2 = \sigma^2 w_i(\theta),$$

where $w_i(\theta)$ are known functions and the parameter $\sigma > 0$ may be unknown. It is easy to see that in this case we can put

$$\gamma_{\text{opt},i}(\theta, \sigma_i) = \gamma_{\text{opt},i}(\theta) \equiv \frac{a_i b_i}{(1 + b_i\theta)^3 w_i(\theta)}.$$

Now, suppose that all conditions of Theorem 2 are satisfied for

$$\gamma_i(\theta) = \gamma_{\text{opt},i}(\theta), \quad c_i = \gamma_{\text{opt},i}(\theta_0), \tag{2.7}$$

where θ_0 is some fixed value of θ. Then the conclusion of Theorem 2 holds; moreover, $d^2(\{\gamma_i(\theta)\}) = d^2_{\text{opt}}$.

Thus, in the case of Example 2 we can recommend using the estimators θ^* and θ^{**} for c_i and $\gamma_i(\theta)$ in (2.7). Moreover, the estimator θ^{**} of the second step is asymptotically normal with the asymptotic variance d^2_{opt}.

Example 3 Suppose that $\sigma_i^2 = \sigma^2$, where $\sigma > 0$ is an unknown parameter. By analogy with Example 2, in this case we can recommend using the estimators θ^* and θ^{**} for $c_i = a_i b_i$ and $\gamma_i(\theta) = \gamma_{\mathrm{opt},i}(\theta) \equiv a_i b_i/(1+b_i\theta)^3$.

Remark 4. If the exact form of the variance σ_i is unknown then we cannot find $\gamma_{\mathrm{opt},i}(\theta,\sigma_i)$ and construct the estimator θ^{**} for $\gamma_i(\theta) = \gamma_{\mathrm{opt},i}(\theta,\sigma_i)$. In this case we can recommend taking $\gamma_i(\theta)$ to be functions that may be assumed to "differ slightly" from the unknown functions $\gamma_{\mathrm{opt},i}(\theta,\sigma_i)$.

Remark 5. It is easy to verify that

$$1 \le \frac{d^2(\{\gamma_i(\theta)\})}{d^2_{\mathrm{opt}}} \le \frac{\sup_{i\le N}(\gamma_i(\theta)/\gamma_{\mathrm{opt},i}(\theta,\sigma_i))}{\inf_{i\le N}(\gamma_i(\theta)/\gamma_{\mathrm{opt},i}(\theta,\sigma_i))}; \tag{2.8}$$

i.e., the "better" the chosen functions $\gamma_i(\theta)$ approximate the functions $\gamma_{\mathrm{opt},i}(\theta,\sigma_i)$, the less the asymptotic variance of the corresponding estimator differs from d^2_{opt}. We can treat (2.8) as some stability property for the estimators θ^{**} as functionals depending on the functions $\gamma_i(\theta)$.

In constructing the confidence intervals and test of hypotheses, it would be more convenient to have analogs of Theorems 1 and 2 in which the parameters $d(\{c_i\})$ and $d(\{\gamma_i(\theta)\})$ are replaced with some their estimators. We give assertions which possess these properties. Put

$$\begin{gathered}\beta_i^* = (1+b_i\theta^*)X_i - a_i, d^* = \Big(\sum c_i^2\beta_i^{*2}\Big)^{1/2}\Big/\sum c_i b_i X_i,\\ d^{**} = \Big(\sum \gamma_i^2(\theta^*)\Big)\beta_i^{*2})^{1/2}\Big/\sum \gamma_i(\theta^*) b_i X_i.\end{gathered} \tag{2.9}$$

Theorem 3. *Suppose that the conditions of Theorem 1 are satisfied. Then*

$$\sup_x |\mathbf{P}((\theta^* - \theta)/d^* < x) - \Phi(x)| \to 0.$$

Theorem 4. *Suppose that the conditions of Theorem 2 are satisfied. Then*

$$\sup_x |\mathbf{P}((\theta^{**} - \theta)/d^{**} < x) - \Phi(x)| \to 0.$$

3. Some Generalizations

Below we consider a more general problem in which $a_i = a_i^{(N)}$, $b_i = b_i^{(N)}$, $\sigma_i = \sigma_i^N$ and $X_i = X_i^{(N)}$, with the superscript emphasizing that the variables may depend on the number N of observations. In order to keep our notation reasonable, we will omit the superscript (N) of the values a_i, b_i, σ_i, and X_i.

We need the following notations:

$$\alpha_i = a_i b_i/(1+b_i\theta), \quad \beta_i = (1+b_i\theta)\sigma_i, \quad \gamma_i = \gamma_i(\theta), \quad K_i = K_i(\theta), \tag{3.1}$$

$$\begin{gathered}A_c = \sum c_i\alpha_i, \quad B_c^2 = \sum c_i^2\beta_i^2, \quad d_c = B_c/A_c,\\ A_\gamma = \sum \gamma_i\alpha_i, \quad B_\gamma^2 = \sum \gamma_i^2\beta_i^{;} \quad d_\gamma = B_\gamma/A_\gamma,\end{gathered} \tag{3.2}$$

$$\hat d_c = B_c\Big/\sum c_i b_i X_i, \quad \hat d_\gamma = B_\gamma\Big/\sum \gamma_i b_i X_i. \tag{3.3}$$

It is easy to see that in this case $d_c = d(\{c_i\})$ and $d_\gamma = d(\{\gamma_i\})$.

Theorem 5. *Suppose that the condition*

$$\max_{k\le N} c_k^2\beta_k^2\Big/B_c^2 \to 0 \tag{3.4}$$

is satisfied. Then

$$\sup_x |\mathbf{P}((\theta^* - \theta)/\hat{d}_c < x) - \Phi(x)| \to 0.$$

Theorem 6. *Suppose that (3.4) and the condition*

$$\sum c_i^2 b_i^2 \sigma_i^2 / A_c^2 \to 0 \tag{3.5}$$

are satisfied. Then the conclusion of Theorem 1 holds.

Theorem 7. *Suppose that (3.4), (3.5), and*

$$\sum c_i^2 \alpha_i^2 / A_c^2 \to 0 \tag{3.6}$$

are satisfied. Then the conclusion of Theorem 4 holds.

It is easy to see that Theorems 1 and 3 are immediate from Theorems 6 and 7.

We turn to studying the estimator θ^{**}. Studying the properties of this estimator, we always assume the following conditions to be satisfied:

$$d_c \to 0, \tag{3.7}$$

$$\max_{k \le N} \gamma_k^2 \beta_k^2 / B_\gamma^2 \to 0, \tag{3.8}$$

$$d_c^2 \Big(\sum \beta_i^2 K_i^2\Big) / B_\gamma^2 \to 0. \tag{3.9}$$

Theorem 8. *Suppose that (3.7)–(3.9) are satisfied. Then*

$$\sup_x |\mathbf{P}((\theta^* - \theta)/\hat{d}_\gamma < x) - \Phi(x)| \to 0.$$

Theorem 9. *Suppose that the conditions of Theorem 8 are satisfied and*

$$\sum \gamma_i^2 b_i^2 \sigma_i^2 / A_\gamma^2 \to 0, \tag{3.10}$$

$$d_c^2 \Big(\sum b_i^2 \sigma_i^2 K_i^2\Big) / A_\gamma^2 \to 0, \tag{3.11}$$

$$d_c \Big(\sum \alpha_i K_i\Big) / A_\gamma \to 0. \tag{3.12}$$

Then all conclusions of Theorem 2 are valid.

Theorem 10. *Suppose that the conditions of Theorem 8 are satisfied and*

$$d_c^4 \sum K_i^2 \big(\alpha_i^2 + b_i^2 \sigma_i^2\big) / B_\gamma^2 \to 0, \tag{3.13}$$

$$d_c^2 \sum \gamma_i^2 \big(\alpha_i^2 + b_i^2 \sigma_i^2\big) / B_\gamma^2 \to 0. \tag{3.14}$$

Then the conclusion of Theorem 4 is valid.

It is easy to verify that Theorems 2 and 4 are particular instances of Theorems 9 and 10.

Remark 6. Studying the estimator θ^{**} of the second step, we essentially use the assumption that the estimator θ^* of the first step is consistent. As we see from (3.18) and (3.2), consistency of this estimator is guaranteed by (3.7). Observe that (3.7) is essential: otherwise the estimator θ^* could satisfy the conditions of Theorem 1 but fail to be consistent. Indeed, suppose that $a_i = c_i = \sigma_i = 1$ and $b_i = 1/i^{1-\varepsilon}$ for $0 < \varepsilon < 1/2$. It is easy to verify that in this case

$$A_c \sim \sum b_i \sim N^\varepsilon/\varepsilon \to \infty, \quad d_c^2 \sim N/A_c^2 \sim \varepsilon^2 N^{1-2\varepsilon} \to \infty.$$

At the same time, all conditions of Theorem 1 are satisfied, because

$$\sum c_i^2 b_i^2 \sigma_i^2 = \sum i^{2\varepsilon-2} < \infty,$$

$$\max_{k\le N} c_k^2\beta_k^2 = (1+\theta)^2 < \infty, \quad \sum c_i^2\beta_i^2 \sim N \to \infty.$$

Remark 7. We present some arguments that enable us to guess the simple form of estimators (1.3) and (1.4). Rewrite (1.1) as

$$(1+b_i\theta)X_i = a_i + \beta_i\xi_i. \tag{3.15}$$

Multiplying (3.15) by c_i and summing the result over i, we come to the following useful identity:

$$\sum c_iX_i + \theta\sum c_ib_iX_i = \sum c_ia_i + \sum c_i\beta_i\xi_i. \tag{3.16}$$

We may suppose that the weighted sum of the errors of ξ_i, the last summand in (3.16), is small as compared with the other sums of positive summands. Therefore, it is natural to discard the last summand in (3.16), substituting the estimator θ^* for the unknown parameter θ in the modified equality. Solving the equation

$$\sum c_iX_i + \theta^*\sum c_ib_iX_i = \sum c_ia_i, \tag{3.17}$$

we find representation (1.3) for the estimator θ^*.

Now, subtracting (3.16) from (3.17) and using (1.1), we arrive at

$$\theta^* - \theta = \frac{-\sum c_i(1+b_i\theta)\sigma_i\xi_i}{\sum c_ib_iX_i} = \frac{-\sum c_i\beta_i\xi_i}{\sum c_i\alpha_i + \sum c_ib_i\sigma_i\xi_i}. \tag{3.18}$$

Representation (3.18) plays a key role in studying the properties of the estimator θ^*.

Observe that, by analogy with (3.18), we have the following representation for the estimator θ^{**}:

$$\theta^{**} - \theta = \frac{\sum \gamma_i(\theta^*)\beta_i\xi_i}{\sum \gamma_i(\theta^*)b_iX_i}. \tag{3.19}$$

Remark 8. Throughout the article, θ is an unknown parameter. Moreover, the values $\{\sigma_i\}$ may be unknown parameters either. Thus, the most of the conditions in all assertions of the article are constraints on the values involving unknown parameters. Clearly, in practical application of these assertions we should check the conditions *for all values of all unknown parameters* (as, for instance, in [3]).

4. PROOFS OF THE PROPERTIES OF THE ESTIMATOR

θ^*

Proof of Theorem 5. Put

$$\beta_{i,N} = c_i\beta_i/B_c, \quad \beta(N) = \max_{k\le N}\beta_{k,N}. \tag{4.1}$$

In this case, (3.18) and (3.3) yield the representation

$$(\theta^* - \theta)/\hat{d}_c = -\sum \beta_{i,N}\xi_i. \tag{4.2}$$

Lemma 1. *If (3.4) is satisfied then*

$$\sup_x \Big|\mathbf{P}\Big(-\sum \beta_{i,N}\xi_i < x\Big) - \Phi(x)\Big| \to 0.$$

Proof. This assertion is a particular instance of the central limit theorem for a scheme of series. Therefore, it suffices (see [4, Chapter 8, Theorem 5]) to verify validity of the condition

$$D_2 \equiv \sum \mathbf{E}\min\{(\beta_{i,N}\xi_i)^2, |\beta_{i,N}\xi_i|^3\} \to 0. \tag{4.3}$$

From the definition (4.1) of $\beta(N)$ we obtain

$$D_2 \le \sum \beta_{i,N}^2 \mathbf{E}\min\{\xi_i^2, \beta(N)|\xi_i|^3\} = \mathbf{E}\min\{\xi_1^2, \beta(N)|\xi_1|^3\}. \tag{4.4}$$

In (4.4) we have used the fact that $\sum \beta_{i,N}^2 = 1$ by (4.1). Now, to derive (4.3) from (4.4), it suffices to observe that $\beta(N) \to 0$, because $\beta(N)$ coincides with the left-hand side of (3.4) by definition.

The claim of Theorem 5 is immediate from (4.2) and Lemma 1.

We turn to proving Theorem 6. We use the representation

$$\frac{\theta^* - \theta}{d_c} = \frac{-\sum \beta_{i,N}\xi_i}{1 + \sum c_i b_i \sigma_i \xi_i / A_c} \tag{4.5}$$

which follows from (2.2) and (3.18).

Lemma 2. *If (3.5) is satisfied then* $\sum c_i b_i \sigma_i \xi_i / A_c \xrightarrow{p} 0$.

This assertion follows from Chebyshev's inequality, since

$$\mathbf{E}\Bigl(\sum c_i b_i \sigma_i \xi_i / A_c\Bigr) = 0$$

and the variance of this expression coincides with the left-hand side of (3.5).

The claim of Theorem 6 is immediate from (4.5), Lemma 2, and Theorem 5.

Now, we turn to proving Theorem 7. We first prove two auxiliary assertions:

Lemma 3. *Suppose that (3.4) is satisfied. Then*

$$\sum c_i^2 \beta_i^2 \xi_i^2 \Big/ \sum c_i^2 \beta_i^2 \xrightarrow{p} 1. \tag{4.6}$$

Proof. Using (4.1), we can rewrite the claim of the lemma as

$$\sum \beta_{i,N}^2 \bigl(\xi_i^2 - 1\bigr) \xrightarrow{p} 0, \quad \mathbf{E}\bigl(\xi_i^2 - 1\bigr) = 0. \tag{4.7}$$

However, the convergence in (4.7) is a particular instance of the law of large numbers for a scheme of series (see [4, Chapter 8, Theorem 3]). Therefore, to validate (4.7), it suffices to show that the following condition is satisfied:

$$D_1 \equiv \sum \mathbf{E}\min\bigl\{\beta_{i,N}^4 \bigl(\xi_i^2 - 1\bigr)^2; \bigl|\beta_{i,N}^2\bigl(\xi_i^2 - 1\bigr)\bigr|\bigr\} \to 0. \tag{4.8}$$

We have

$$D_1 \le \sum \beta_{i,N}^2 \mathbf{E}\min\bigl\{\beta^2(N)\bigl(\xi_i^2 - 1\bigr)^2; \bigl|\xi_i^2 - 1\bigr|\bigr\} = \mathbf{E}\min\bigl\{\beta^2(N)\bigl(\xi_1^2 - 1\bigr)^2; \bigl|\xi_1^2 - 1\bigr|\bigr\}. \tag{4.9}$$

Deriving (4.9), we have used the fact that $\sum \beta_{i,N}^2 = 1$ in view of (4.1). Since $\beta(N) \to 0$ by (3.4), from (4.9) we now obtain (4.8), which proves the sought assertion (4.7).

Denote

$$\delta^* = \Bigl(\sum c_i^2 \beta_i^{*2}\Bigr)^{1/2} - \Bigl(\sum c_i^2 \beta_i^2 \xi_i^2\Bigr)^{1/2}, \quad \delta_c = \Bigl(\sum c_i^2 \beta_i^2 \xi_i^2\Bigr)^{1/2} - \Bigl(\sum c_i^2 \beta_i^2\Bigr)^{1/2}. \tag{4.10}$$

Lemma 4. *Under the conditions of Theorem 7,*

$$\delta_c / B_c \xrightarrow{p} 0, \quad \delta^* / B_c \xrightarrow{p} 0.$$

Proof. The first claim of the lemma following from Lemma 3, it suffices to prove the second. Observe that

$$\beta_i^* - \beta_i \xi_i = (1 + b_i\theta^*)X_i - a_i - (1 + b_i\theta)\sigma_i\xi_i = (\theta^* - \theta)b_i X_i.$$

Furthermore as soon as, the root of the sum of squares possesses all properties of a norm, we have

$$|\delta^*| \le \Big(\sum c_i^2(\beta_i^* - \beta_i\xi_i)^2\Big)^{1/2} = |\theta^* - \theta|\delta_{0c},$$

where $\delta_{0c} = \big(\sum c_i^2 b_i^2 X_i^2\big)^{1/2}$. Therefore

$$\frac{|\delta^*|}{B_c} \le \frac{|\theta^* - \theta|}{d_c}\frac{\delta_{0c}}{A_c}. \tag{4.11}$$

Since $\mathbf{E}\delta_{0c}^2 = \sum c_i^2\alpha_i^2 + \sum c_i^2 b_i^2\sigma_i^2$, by (3.5) and (3.6) the second factor in (4.11) vanishes in probability. Using this fact and asymptotic normality of the estimator θ^*, from (4.11) we easily deduce the second claim of the lemma.

Now, we complete the proof of Theorem 7. Using (3.18) and the notations (3.2), (4.1), and (4.10), we obtain

$$\frac{\theta^* - \theta}{d^*} = \frac{-\sum c_i\beta_i\xi_i}{\big(\sum c_i^2\beta_i^{*2}\big)^{1/2}} = \frac{-\sum \beta_{i,N}\xi_i}{1 + \delta^*/B_c + \delta_c/B_c}. \tag{4.12}$$

Now, Theorem 7 is immediate from (4.12) and Lemmas 1 and 4.

5. Proof of Theorem 8

Fix $\theta > 0$, preserving the above notations. Put

$$q_i = -c_i\sigma_i/A_c, \quad r_i = c_i b_i\sigma_i/A_c, \quad Z_1 = -\sum q_i\xi_i, \quad Z_2 = \sum r_i\xi_i. \tag{5.1}$$

Then by (3.18)

$$\theta^* = (\theta + Z_1)/(1 + Z_2). \tag{5.2}$$

Introduce the random variable

$$\tilde\theta = \min\{2\theta, \theta + Z_1\}/\max\{1/2, 1 + Z_2\}. \tag{5.3}$$

Lemma 5. *If (3.7) is satisfied then*

$$\mathbf{P}(\tilde\theta \ne \theta^*) \to 0.$$

Proof. By (1.2) and (5.1) we have $\mathbf{E}Z_1 = \mathbf{E}Z_2 = 0$; therefore,

$$\mathbf{E}(Z_1)^2 = \sum c_i^2\sigma_i^2/A_c^2 \le B_c^2/A_c^2 = d_c^2 \to 0, \tag{5.4}$$

$$\mathbf{E}(Z_2)^2 = \sum c_i^2 b_i^2\sigma_i^2/A_c^2 \le B_c^2/A_c^2\theta^2 = d_c^2/\theta^2 \to 0 \tag{5.5}$$

by (3.7). We infer in particular that $Z_1 \xrightarrow{p} 0$ and $Z_2 \xrightarrow{p} 0$. Thus, to complete the proof of Lemma 5, we need to compare (5.2) and (5.3), observing that

$$\mathbf{P}(\tilde\theta \ne \theta^*) \le \mathbf{P}(Z_2 < -1/2) + \mathbf{P}(Z_1 > \theta) \to 0.$$

Put

$$\tilde\gamma_i = \gamma_i(\tilde\theta), \quad Z_3 = \sum(\tilde\gamma_i - \gamma_i)\beta_i\xi_i, \quad \tilde d_\gamma = B_\gamma\Big/\sum\tilde\gamma_i b_i X_i, \tag{5.6}$$

$$\tilde\theta^{**} = \sum\tilde\gamma_i(a_i - X_i)\Big/\sum\tilde\gamma_i b_i X_i. \tag{5.7}$$

From (1.4), (5.6), (5.7), and Lemma 5 we immediately see that

$$\mathbf{P}(\tilde\theta^{**} \ne \theta^{**}) \to 0, \quad \mathbf{P}(\tilde d_\gamma \ne \hat d_\gamma) \to 0. \tag{5.8}$$

Using (1.4), (3.19), (5.6), and (5.7), we obtain

$$(\tilde{\theta}^{**}-\theta)/\tilde{d}_\gamma = -\sum \tilde{\gamma}_i\beta_i\xi_i/B_\gamma = -\sum \gamma_i\beta_i\xi_i/B_\gamma + Z_3/B_\gamma. \tag{5.9}$$

Formula (5.9) is central to the proof of Theorem 8. Also, we need the notations

$$\tilde{\theta}_j = \frac{\min\{2\theta, \theta + \sum\limits_{i\neq j} q_i\xi_i\}}{\max\{1/2, 1 + \sum\limits_{i\neq j} r_i\xi_i\}}, \quad \tilde{\theta}_{jk} = \frac{\min\{2\theta, \theta + \sum\limits_{i\neq j,k} q_i\xi_i\}}{\max\{1/2, 1 + \sum\limits_{i\neq j,k} r_i\xi_i\}}.$$

Lemma 6. *For each j, the value $\tilde{\theta}_j$ is independent of ξ_j and*

$$|\tilde{\theta}_j - \tilde{\theta}| \le \widetilde{K}_j|\xi_j| \quad \textit{for} \quad \widetilde{K}_j = 8(|q_j| + \theta r_j) = 8c_j\beta_j/A_c.$$

Proof. The independence follows from the definition of $\tilde{\theta}_j$. Now, fix a number j and all values ξ_i for $i \neq j$. Put

$$q = \theta + \sum_{i\neq j} q_i\xi_i, \quad r = 1 + \sum_{i\neq j} r_i\xi_i,$$

$$f_1(\xi_j) = \min\{2\theta, q + q_j\xi_j\}, \quad f_2(\xi_j) = \max\{1/2, 1 + r + r_j\xi_j\}.$$

With these notations, we have $\tilde{\theta} = f_1(\xi_j)/f_2(\xi_j)$ and $\tilde{\theta}_j = f_1(0)/f_2(0)$. Therefore,

$$\tilde{\theta} - \tilde{\theta}_j = \frac{f_1(\xi_j) - f_1(0)}{f_2(\xi_j)} + f_1(0)\frac{f_2(0) - f_2(\xi_j)}{f_2(\xi_j)f_2(0)}. \tag{5.10}$$

It is clear that

$$|f_1(\xi_j) - f_1(0)| \le |q_j||\xi_j|, \quad |f_2(\xi_j) - f_2(0)| \le r_j|\xi_j|, \quad |f_1(\cdot)| \le 2\theta, \quad |f_2(\cdot)| \ge 1/2.$$

Inserting these relations in (5.10), we obtain

$$|\tilde{\theta} - \tilde{\theta}_j| \le \frac{|q_j||\xi_j|}{1/2} + 2\theta\frac{r_j|\xi_j|}{1/4} \le 8(|q_j| + \theta r_j)|\xi_j|.$$

Lemma 7. *For all $j \neq k$, the value $\tilde{\theta}_{jk}$ is independent of ξ_j and ξ_k, and*

$$|\tilde{\theta}_{jk} - \tilde{\theta}_k| \le \widetilde{K}_j|\xi_j|.$$

Proof. The proof of the lemma repeats that of Lemma 6 on taking ξ_k to be identically zero.

Lemma 8. *The following inequality is valid:*

$$\mathbf{E}|\tilde{\theta} - \theta|^2 \le 16d_c^2.$$

Proof. By analogy with Lemma 6, we put

$$f_1(Z_1) = \min\{2\theta, \theta + Z_1\}, \quad f_2(Z_2) = \max\{1/2, 1 + Z_2\}.$$

It is clear that $f_1(0) = \theta$, $f_2(0) = 1$, $|f_2(Z_2)| \ge 1/2$, and

$$|f_1(Z_1) - f_1(0)| \le |Z_1|, \quad |f_2(Z_2) - f_2(0)| \le |Z_2|.$$

Therefore,

$$|\tilde\theta-\theta| = \left|\frac{f_1(Z_1)}{f_2(Z_2)}-\frac{f_1(0)}{f_2(0)}\right| = \left|\frac{f_1(Z_1)-f_1(0)}{f_2(Z_2)}+f_1(0)\frac{f_2(0)-f_2(Z_2)}{f_2(Z_2)f_2(0)}\right|$$
$$\le \frac{|Z_1|}{1/2}+\theta\frac{|Z_2|}{1/2} = 2|Z_1|+2\theta|Z_2|.$$

Consequently,

$$\mathbf{E}|\tilde\theta-\theta|^2 \le 2\mathbf{E}(2|Z_1|)^2+2\mathbf{E}(2\theta|Z_2|)^2.$$

Inserting the estimates of (5.4) and (5.5) in the last inequality, we deduce the claim.

Lemma 9. *For all* i *we have*

$$\mathbf{E}|\tilde\theta_i-\theta|^2 \le 16d_c^2.$$

Proof. The proof of the lemma repeats verbatim that of Lemma 8 on taking ξ_i to be identically zero.

Put $\tilde\gamma_{ii}=\gamma_i(\tilde\theta_i)$.

Lemma 10. *The following inequality is valid:*

$$\Delta_1 \equiv \mathbf{E}\Big|\sum\beta_i(\tilde\gamma_i-\tilde\gamma_{ii})\xi_i\Big| \le 8d_c\Big(\sum\beta_i^2K_i^2\Big)^{1/2}.$$

Proof. By (2.3) and Lemma 6, we have

$$|\tilde\gamma_i-\tilde\gamma_{ii}| \le K_i|\tilde\theta-\tilde\theta_i| \le K_i\widetilde K_i|\xi_i|.$$

Consequently,

$$\Delta_1 \le \mathbf{E}\sum\beta_iK_i\widetilde K_i\xi_i^2 = \sum\beta_iK_i\widetilde K_i. \tag{5.11}$$

Using the definition of the constants $\{\widetilde K_i\}$, we now obtain

$$\Big(\sum\beta_iK_i\widetilde K_i\Big)^2 \le \sum\beta_i^2K_i^2\sum\widetilde K_i^2 = (8d_c)^2\sum\beta_i^2K_i^2. \tag{5.12}$$

Inequalities (5.11) and (5.12) yield the claim of Lemma 10.

Lemma 11. *The following inequality is valid:*

$$\Delta_2 \equiv \mathbf{E}\Big(\sum\beta_i(\tilde\gamma_{ii}-\gamma_i)\xi_i\Big)^2 \le 80d_c^2\Big(\sum\beta_i^2K_i^2\Big).$$

Proof. Put $\delta_{ii}=\tilde\gamma_{ii}-\gamma_i$. Clearly, δ_{ii} is independent of ξ_i. Therefore,

$$\begin{aligned}\mathbf{E}\Big(\sum\beta_i\delta_{ii}\xi_i\Big)^2 &= \sum\beta_i^2\mathbf{E}\delta_{ii}^2\xi_i^2+\sum_{i\ne j}\beta_i\beta_j\mathbf{E}\delta_{ii}\xi_i\delta_{jj}\xi_j\\ &= \sum\beta_i^2\mathbf{E}\delta_{ii}^2+\sum_{i\ne j}\beta_i\beta_j\mathbf{E}\delta_{ii}\xi_i\delta_{jj}\xi_j.\end{aligned} \tag{5.13}$$

Using (2.3) and Lemma 9, we obtain

$$\sum\beta_i^2\mathbf{E}\delta_{ii}^2 \le \sum\beta_i^2\mathbf{E}K_i^2|\tilde\theta_i-\theta|^2 \le 16d_c^2\Big(\sum\beta_i^2K_i^2\Big). \tag{5.14}$$

Denote $\tilde\gamma_{iij}=\gamma_i(\tilde\theta_{ij})$, $\delta_{iij}=\tilde\gamma_{iij}-\gamma_i$, and $\tilde\delta_{iij}=\tilde\gamma_{ii}-\tilde\gamma_{iij}$. Then

$$\delta_{ii}=\delta_{iij}+\tilde\delta_{iij}; \tag{5.15}$$

moreover, $\tilde{\delta}_{iij}$ is independent of ξ_i and by Lemma 7 and (2.3)

$$|\tilde{\delta}_{iij}| \le K_i|\tilde{\theta}_i - \tilde{\theta}_{ij}| \le K_i\widetilde{K}_j|\xi_j|.$$

Inserting (5.15) in the second summand of (5.13), we obtain

$$\begin{aligned}\sum_{i\neq j}\beta_i\beta_j\mathbf{E}\delta_{ii}\xi_i\delta_{jj}\xi_j &= \sum_{i\neq j}\beta_i\beta_j\mathbf{E}(\delta_{iij}+\tilde{\delta}_{iij})\xi_i(\delta_{jji}+\tilde{\delta}_{jji})\xi_j\\ &= \sum_{i\neq j}\beta_i\beta_j\mathbf{E}\delta_{iij}\delta_{jji}\xi_i\xi_j + \sum_{i\neq j}\beta_i\beta_j\mathbf{E}\delta_{iij}\tilde{\delta}_{jji}\xi_i\xi_j\\ &+ \sum_{i\neq j}\beta_i\beta_j\mathbf{E}\delta_{jji}\tilde{\delta}_{iij}\xi_i\xi_j + \sum_{i\neq j}\beta_i\beta_j\mathbf{E}\tilde{\delta}_{iij}\tilde{\delta}_{jji}\xi_i\xi_j\\ &\le 0+0+0+\sum_{i\neq j}\beta_i\beta_j\mathbf{E}K_i\widetilde{K}_iK_j\widetilde{K}_j\xi_i^2\xi_j^2\\ &\le \Bigl(\sum\beta_iK_i\widetilde{K}_i\Bigr)^2 \le 64d_c^2\Bigl(\sum\beta_i^2K_i^2\Bigr).\end{aligned} \tag{5.16}$$

Deriving the last relation, we have used (5.12).

The claim of the lemma ensues from (5.14), (5.16), and (5.13).

Lemma 12. *The following inequality is valid:*

$$\Delta_3 \equiv \mathbf{E}|Z_3| \le 17d_c\Bigl(\sum\beta_i^2K_i^2\Bigr)^{1/2}.$$

To prove this assertion, it suffices to observe that $\Delta_3 \le \Delta_1 + \Delta_2^{1/2}$ and use Lemmas 10 and 11.

Lemma 13. *If (3.8) is satisfied then*

$$\sup_x\Bigl|\mathbf{P}\Bigl(-\sum\gamma_i\beta_i\xi_i/B_\gamma < x\Bigr) - \Phi(x)\Bigr| \to 0. \tag{5.17}$$

To obtain this convergence, it suffices to repeat the derivation of Lemma 2 for $\beta_{i,N} = -\gamma_i\beta_i/B_\gamma$.

We now complete the proof of Theorem 8. It follows from Lemma 12 and (3.9) that

$$Z_3/B_\gamma \xrightarrow{p} 0. \tag{5.18}$$

Therefore, the claim of Theorem 8 ensues from (5.9), (5.18), and Lemma 13.

6. PROOFS OF THEOREMS 9 AND 10

Denote

$$Z_4 = \sum(\tilde{\gamma}_i - \gamma_i)\alpha_i, \quad Z_5 = \sum\gamma_ib_i\sigma_i\xi_i, \quad Z_6 = \sum(\tilde{\gamma}_i-\gamma_i)b_i\sigma_i\xi_i. \tag{6.1}$$

Using (1.4), (3.19), and the notations (3.2), (5.6), and (5.7), we obtain

$$\frac{\tilde{\theta}^{**}-\theta}{d(\{\gamma_i\})} = \frac{-\sum\tilde{\gamma}_i\beta_i\xi_i}{\sum\tilde{\gamma}_ib_iX_i}\cdot\frac{A_\gamma}{B_\gamma} = \frac{-\sum\gamma_i\beta_i\xi_i/B_\gamma + Z_3/B_\gamma}{1+(Z_4+Z_5+Z_6)/A_\gamma}. \tag{6.2}$$

Representation (6.2) plays a key role in the proof of Theorem 9.

Lemma 14. *The following estimate holds:*

$$\Delta_6 \equiv \mathbf{E}|Z_6| \le 17d_c\left(\sum b_i^2\sigma_i^2K_i^2\right)^{1/2}.$$

To derive this inequality, it suffices to repeat the proof of Lemmas 10–12 with $b_i\sigma_i$ substituted for β_i.

Lemma 15. *The following estimate is valid:*

$$\Delta_4 \equiv \mathbf{E}|Z_4| \le 4d_c \left(\sum \alpha_i K_i\right).$$

Proof. By (2.3) we have $|\tilde\gamma_i - \gamma_i| \le K_i|\tilde\theta - \theta|$. Using Lemma 8, we obtain

$$\mathbf{E}|\tilde\gamma_i - \gamma_i| \le K_i(\mathbf{E}(\tilde\theta-\theta)^2)^{1/2} \le (16)^{1/2} d_c K_i.$$

Inserting this inequality in the definition (6.1) of Z_4, we arrive at the claim of the lemma.

Lemma 16. *The following estimate is valid:*

$$\Delta_5 \equiv \mathbf{E}|Z_5| \le \left(\sum \gamma_i^2 b_i^2 \sigma_i^2\right)^{1/2}.$$

To prove this inequality, it suffices to note that $\mathbf{E}Z_5 = 0$ and $\mathbf{D}Z_5 = \sum \gamma_i^2 b_i^2 \sigma_i^2$ in view of (1.2) and (6.1).

Now, we complete the proof of Theorem 9. It follows from Lemmas 14–16 and (3.10)–(3.12) that $\mathbf{E}|Z_4+Z_5+Z_6|/A_\gamma \to 0$; therefore,

$$(Z_4+Z_5+Z_6)/A_\gamma \xrightarrow{p} 0. \tag{6.3}$$

The claim of the theorem ensues now from (6.2), (6.3), (5.18), and Lemma 13.

We turn to proving Theorem 10. Put

$$\tilde\beta_i = (1+b_i\tilde\theta)X_i - a_i, \quad \tilde d^{**} = \left(\sum \tilde\gamma_i^2\tilde\beta_i^2\right)^{1/2} \Big/ \sum \tilde\gamma_i b_i X_i, \tag{6.4}$$

$$\delta = \left(\sum \tilde\gamma_i^2\tilde\beta_i^2\right)^{1/2} - \left(\sum \gamma_i^2\beta_i^2\xi_i^2\right)^{1/2}, \quad \delta_0 = \left(\sum \gamma_i^2\beta_i^2\xi_i^2\right)^{1/2} - \left(\sum \gamma_i^2\beta_i^2\right)^{1/2} \tag{6.5}$$

and note that

$$\mathbf{P}(\tilde d^{**} \ne d^{**}) \to 0 \tag{6.6}$$

by Lemma 5 and (2.9), (5.6), and (6.4). Using (3.19) and the notations (1.4), (5.6), (5.7), (6.4), and (6.5), we easily obtain the formula

$$\frac{\tilde\theta^{**}-\theta}{\tilde d^{**}} = \frac{-\sum \tilde\gamma_i\beta_i\xi_i}{\left(\sum \tilde\gamma_i^2\tilde\beta_i^2\right)^{1/2}} = \frac{-\sum \gamma_i\beta_i\xi_i/B_\gamma + Z_3/B_\gamma}{1+(\delta+\delta_0)/B_\gamma}. \tag{6.7}$$

This formula plays a key role in the proof of Theorem 10.

Denote

$$\delta_{0\beta} = \left(\sum K_i^2\beta_i^2\xi_i^2\right)^{1/2}, \quad \delta_{0\gamma} = \left(\sum \gamma_i^2 b_i^2 X_i^2\right)^{1/2}, \quad \delta_{0K} = \left(\sum K_i^2 b_i^2 X_i^2\right)^{1/2}. \tag{6.8}$$

Lemma 17. *The following inequality is valid:*

$$|\delta| \le (\tilde\theta-\theta)^2\delta_{0K} + |\tilde\theta-\theta|(\delta_{0\gamma}+\delta_{0\beta}).$$

Proof. From the properties of a norm we obtain

$$|\delta| \le \Big(\sum(\tilde\gamma_i\tilde\beta_i - \gamma_i\beta_i\xi_i)^2\Big)^{1/2}$$
$$= \Big(\sum((\tilde\gamma_i-\gamma_i)(\tilde\beta_i-\beta_i\xi_i)+\gamma_i(\tilde\beta_i-\beta_i\xi_i)+\beta_i\xi_i(\tilde\gamma_i-\gamma_i))^2\Big)^{1/2} \le \delta_{\beta\gamma}+\delta_\beta+\delta_\gamma, \tag{6.9}$$

where

$$\delta_{\beta\gamma} = \Big(\sum(\tilde\gamma_i-\gamma_i)^2(\tilde\beta_i-\beta_i\xi_i)^2\Big)^{1/2},$$
$$\delta_\beta = \Big(\sum\gamma_i^2(\tilde\beta_i-\beta_i\xi_i)^2\Big)^{1/2}, \quad \delta_\gamma = \Big(\sum(\tilde\gamma_i-\gamma_i)^2\beta_i^2\xi_i^2\Big)^{1/2}.$$

Using the identity

$$\tilde\beta_i - \beta_i\xi_i = (\tilde\theta-\theta)(\alpha_i+b_i\sigma_i\xi_i) = (\tilde\theta-\theta)b_iX_i$$

and (2.3), we easily infer that

$$\delta_\beta = |\tilde\theta-\theta|\delta_{0\gamma}, \quad \delta_\gamma \le |\tilde\theta-\theta|\delta_{0\beta}, \quad \delta_{\beta\gamma} \le |\tilde\theta-\theta|^2\delta_{0K}. \tag{6.10}$$

From (6.9) and (6.10) we obtain the claim of the lemma.

Lemma 18. *If the conditions of Theorem 10 are satisfied then* $\delta/B_\gamma \xrightarrow{p} 0$.

Proof. Since $(|u|+|v|)^{1/2} \le |u|^{1/2}+|v|^{1/2}$, from Schwarz's inequality and Lemma 17 we find that

$$\mathbf{E}|\delta|^{1/2} \le (\mathbf{E}(\tilde\theta-\theta)^2)^{1/2}\big(\mathbf{E}\delta_{0K}^2\big)^{1/4} + (\mathbf{E}(\tilde\theta-\theta)^2)^{1/4}\big(\big(\mathbf{E}\delta_{0\gamma}^2\big)^{1/4}+\big(\mathbf{E}\delta_{0\beta}^2\big)^{1/4}\big). \tag{6.11}$$

Using (6.8), from the above relation and Lemma 8 we derive

$$\mathbf{E}|\delta|^{1/2} \le \big(16d_c^2\big)^{1/2}\Big(\sum K_i^2\big(\alpha_i^2+b_i^2\sigma_i^2\big)\Big)^{1/4}$$
$$+\big(16d_c^2\big)^{1/4}\Big(\sum\gamma_i^2\big(\alpha_i^2+b_i^2\sigma_i^2\big)\Big)^{1/4} + \big(16d_c^2\big)^{1/4}\Big(\sum K_i^2\beta_i^2\Big)^{1/4}.$$

Dividing the so-obtained expression by $B_\gamma^{1/2}$ and applying (3.13), (3.14), and (3.9), we easily see that $\mathbf{E}(\delta/B_\gamma)^{1/2} \to 0$; whence the claim of the lemma follows.

Lemma 19. *If (3.8) is satisfied then* $\delta_0/B_\gamma \xrightarrow{p} 0$.

To prove this assertion, we have to repeat the derivation of Lemma 3 with the values c_i replaced by γ_i.

We now finish the proof of Theorem 10. According to Lemmas 12 and 13, the distribution of the numerator in (6.7) converges to the standard normal distribution. The denominator in (6.7) converges in probability to 1 by Lemmas 18 and 19. To complete the proof, it remains to observe that $\mathbf{P}(\tilde\theta^{**} \ne \theta^{**}) \to 0$ and $\mathbf{P}(\tilde d^{**} \ne d^{**}) \to 0$ in view of (5.8), (6.6) and (3.7).

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